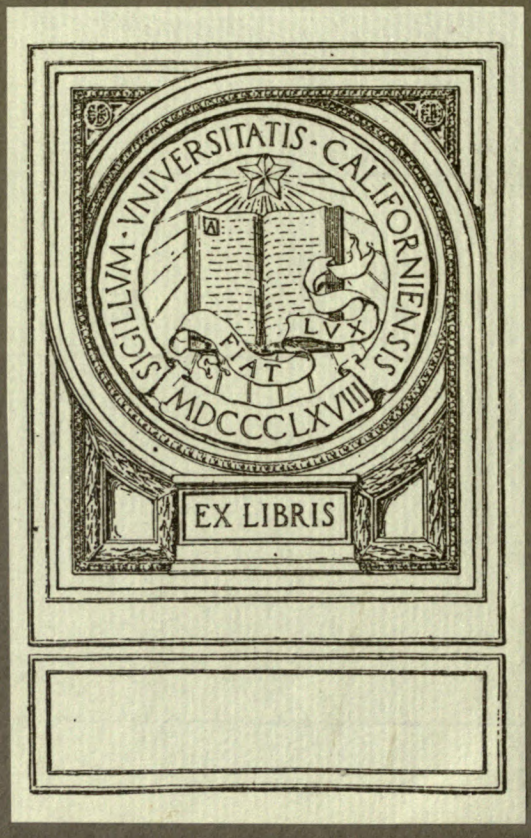
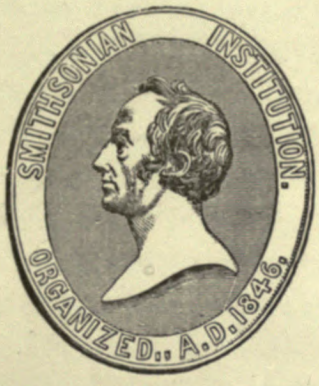

SIGILLVM · VNIVERSITATIS · CALIFORNIENSIS
LVX
FIAT
MDCCCLXVIII
EX LIBRIS

SMITHSONIAN

CONTRIBUTIONS TO KNOWLEDGE.

VOL. XIX.

UNIV. OF CALIFORNIA



EVERY MAN IS A VALUABLE MEMBER OF SOCIETY, WHO, BY HIS OBSERVATIONS, RESEARCHES, AND EXPERIMENTS, PROCURES KNOWLEDGE FOR MEN.—SMITHSON.

CITY OF WASHINGTON:
PUBLISHED BY THE SMITHSONIAN INSTITUTION.
MDCCCLXXIV.

Q11
S68
v.19

UNIV. OF CALIFORNIA

8947

COLLINS, PRINTER,
705 JAYNE ST.

ADVERTISEMENT.

THIS volume forms the nineteenth of a series, composed of original memoirs on different branches of knowledge, published at the expense, and under the direction, of the Smithsonian Institution. The publication of this series forms part of a general plan adopted for carrying into effect the benevolent intentions of JAMES SMITHSON, Esq., of England. This gentleman left his property in trust to the United States of America, to found, at Washington, an institution which should bear his own name, and have for its objects the "*increase* and *diffusion* of knowledge among men." This trust was accepted by the Government of the United States, and an Act of Congress was passed August 10, 1846, constituting the President and the other principal executive officers of the general government, the Chief Justice of the Supreme Court, the Mayor of Washington, and such other persons as they might elect honorary members, an establishment under the name of the "SMITHSONIAN INSTITUTION FOR THE INCREASE AND DIFFUSION OF KNOWLEDGE AMONG MEN." The members and honorary members of this establishment are to hold stated and special meetings for the supervision of the affairs of the Institution, and for the advice and instruction of a Board of Regents, to whom the financial and other affairs are intrusted.

The Board of Regents consists of three members *ex officio* of the establishment, namely, the Vice-President of the United States, the Chief Justice of the Supreme Court, and the Mayor of Washington, together with twelve other members, three of whom are appointed by the Senate from its own body, three by the House of Representatives from its members, and six persons appointed by a joint resolution of both houses. To this Board is given the power of electing a Secretary and other officers, for conducting the active operations of the Institution.

To carry into effect the purposes of the testator, the plan of organization should evidently embrace two objects: one, the increase of knowledge by the addition of new truths to the existing stock; the other, the diffusion of knowledge, thus increased, among men. No restriction is made in favor of any kind of knowledge; and, hence, each branch is entitled to, and should receive, a share of attention.

The Act of Congress, establishing the Institution, directs, as a part of the plan of organization, the formation of a Library, a Museum, and a Gallery of Art, together with provisions for physical research and popular lectures, while it leaves to the Regents the power of adopting such other parts of an organization as they may deem best suited to promote the objects of the bequest.

After much deliberation, the Regents resolved to divide the annual income into two parts—one part to be devoted to the increase and diffusion of knowledge by means of original research and publications—the other part of the income to be applied in accordance with the requirements of the Act of Congress, to the gradual formation of a Library, a Museum, and a Gallery of Art.

The following are the details of the parts of the general plan of organization provisionally adopted at the meeting of the Regents, Dec. 8, 1847.

DETAILS OF THE FIRST PART OF THE PLAN.

I. To increase Knowledge.—*It is proposed to stimulate research, by offering rewards for original memoirs on all subjects of investigation.*

1. The memoirs thus obtained, to be published in a series of volumes, in a quarto form, and entitled "Smithsonian Contributions to Knowledge."

2. No memoir, on subjects of physical science, to be accepted for publication, which does not furnish a positive addition to human knowledge, resting on original research; and all unverified speculations to be rejected.

3. Each memoir presented to the Institution, to be submitted for examination to a commission of persons of reputation for learning in the branch to which the memoir pertains; and to be accepted for publication only in case the report of this commission is favorable.

4. The commission to be chosen by the officers of the Institution, and the name of the author, as far as practicable, concealed, unless a favorable decision be made.

5. The volumes of the memoirs to be exchanged for the Transactions of literary and scientific societies, and copies to be given to all the colleges, and principal libraries, in this country. One part of the remaining copies may be offered for sale; and the other carefully preserved, to form complete sets of the work, to supply the demand from new institutions.

6. An abstract, or popular account, of the contents of these memoirs to be given to the public, through the annual report of the Regents to Congress.

II. To INCREASE KNOWLEDGE.—*It is also proposed to appropriate a portion of the income, annually, to special objects of research, under the direction of suitable persons.*

1. The objects, and the amount appropriated, to be recommended by counsellors of the Institution.

2. Appropriations in different years to different objects; so that, in course of time, each branch of knowledge may receive a share.

3. The results obtained from these appropriations to be published, with the memoirs before mentioned, in the volumes of the Smithsonian Contributions to Knowledge.

4. Examples of objects for which appropriations may be made:—

(1.) System of extended meteorological observations for solving the problem of American storms.

(2.) Explorations in descriptive natural history, and geological, mathematical, and topographical surveys, to collect material for the formation of a Physical Atlas of the United States.

(3.) Solution of experimental problems, such as a new determination of the weight of the earth, of the velocity of electricity, and of light; chemical analyses of soils and plants; collection and publication of articles of science, accumulated in the offices of Government.

(4.) Institution of statistical inquiries with reference to physical, moral, and political subjects.

(5.) Historical researches, and accurate surveys of places celebrated in American history.

(6.) Ethnological researches, particularly with reference to the different races of men in North America; also explorations, and accurate surveys, of the mounds and other remains of the ancient people of our country.

I. To DIFFUSE KNOWLEDGE.—*It is proposed to publish a series of reports, giving an account of the new discoveries in science, and of the changes made from year to year in all branches of knowledge not strictly professional.*

1. Some of these reports may be published annually, others at longer intervals, as the income of the Institution or the changes in the branches of knowledge may indicate.

2. The reports are to be prepared by collaborators, eminent in the different branches of knowledge.

3. Each collaborator to be furnished with the journals and publications, domestic and foreign, necessary to the compilation of his report; to be paid a certain sum for his labors, and to be named on the title-page of the report.

4. The reports to be published in separate parts, so that persons interested in a particular branch, can procure the parts relating to it, without purchasing the whole.

5. These reports may be presented to Congress, for partial distribution, the remaining copies to be given to literary and scientific institutions, and sold to individuals for a moderate price.

The following are some of the subjects which may be embraced in the reports:—

I. PHYSICAL CLASS.

1. Physics, including astronomy, natural philosophy, chemistry, and meteorology.
2. Natural history, including botany, zoology, geology, &c
3. Agriculture.
4. Application of science to arts.

II. MORAL AND POLITICAL CLASS.

5. Ethnology, including particular history, comparative philology, antiquities, &c.
6. Statistics and political economy.
7. Mental and moral philosophy.
8. A survey of the political events of the world; penal reform, &c.

III. LITERATURE AND THE FINE ARTS.

9. Modern literature.
10. The fine arts, and their application to the useful arts.
11. Bibliography.
12. Obituary notices of distinguished individuals.

II. To DIFFUSE KNOWLEDGE.—*It is proposed to publish occasionally separate treatises on subjects of general interest.*

1. These treatises may occasionally consist of valuable memoirs translated from foreign languages, or of articles prepared under the direction of the Institution, or procured by offering premiums for the best exposition of a given subject.

2. The treatises to be submitted to a commission of competent judges, previous to their publication.

DETAILS OF THE SECOND PART OF THE PLAN OF ORGANIZATION.

This part contemplates the formation of a Library, a Museum, and a Gallery of Art.

1. To carry out the plan before described, a library will be required, consisting, 1st, of a complete collection of the transactions and proceedings of all the learned societies of the world; 2d, of the more important current periodical publications, and other works necessary in preparing the periodical reports.

2. The Institution should make special collections, particularly of objects to verify its own publications. Also a collection of instruments of research in all branches of experimental science.

3. With reference to the collection of books, other than those mentioned above, catalogues of all the different libraries in the United States should be procured, in order that the valuable books first purchased may be such as are not to be found elsewhere in the United States.

4. Also catalogues of memoirs, and of books in foreign libraries, and other materials, should be collected, for rendering the Institution a centre of bibliographical knowledge, whence the student may be directed to any work which he may require.

5. It is believed that the collections in natural history will increase by donation, as rapidly as the income of the Institution can make provision for their reception; and, therefore, it will seldom be necessary to purchase any article of this kind.

6. Attempts should be made to procure for the gallery of art, casts of the most celebrated articles of ancient and modern sculpture.

7. The arts may be encouraged by providing a room, free of expense, for the exhibition of the objects of the Art-Union, and other similar societies.

8. A small appropriation should annually be made for models of antiquity, such as those of the remains of ancient temples, &c.

9. The Secretary and his assistants, during the session of Congress, will be required to illustrate new discoveries in science, and to exhibit new objects of art; distinguished individuals should also be invited to give lectures on subjects of general interest.

In accordance with the rules adopted in the programme of organization, each memoir in this volume has been favorably reported on by a Commission appointed

for its examination. It is however impossible, in most cases, to verify the statements of an author; and, therefore, neither the Commission nor the Institution can be responsible for more than the general character of a memoir.

The following rules have been adopted for the distribution of the quarto volumes of the Smithsonian Contributions:—

1. They are to be presented to all learned societies which publish Transactions, and give copies of these, in exchange, to the Institution.

2. Also, to all foreign libraries of the first class, provided they give in exchange their catalogues or other publications, or an equivalent from their duplicate volumes.

3. To all the colleges in actual operation in this country, provided they furnish, in return, meteorological observations, catalogues of their libraries and of their students, and all other publications issued by them relative to their organization and history.

4. To all States and Territories, provided there be given, in return, copies of all documents published under their authority.

5. To all incorporated public libraries in this country, not included in any of the foregoing classes, now containing more than 10,000 volumes; and to smaller libraries, where a whole State or large district would be otherwise unsupplied.

OFFICERS

OF THE

SMITHSONIAN INSTITUTION.

THE PRESIDENT OF THE UNITED STATES,

Ex-officio PRESIDING OFFICER OF THE INSTITUTION.

THE VICE-PRESIDENT OF THE UNITED STATES,

Ex-officio SECOND PRESIDING OFFICER.

MORRISON R. WAITE,

CHANCELLOR OF THE INSTITUTION.

JOSEPH HENRY,

SECRETARY OF THE INSTITUTION.

SPENCER F. BAIRD,

ASSISTANT SECRETARY.

PETER PARKER,
JOHN MACLEAN,
WILLIAM T. SHERMAN,
} EXECUTIVE COMMITTEE.

REGENTS.

HENRY WILSON, ...	*Vice-President of the United States.*
MORRISON R. WAITE, ...	*Chief Justice of the United States.*
HANNIBAL HAMLIN, ...	*Member of the Senate of the United States.*
JOHN W. STEVENSON, ...	" " " " " "
AARON A. SARGENT, ...	" " " " " "
SAMUEL S. COX, ...	*Member of the House of Representatives U. S.*
EBENEZER R. HOAR, ...	" " " " " "
GERRY W. HAZLETON, ...	" " " " " "
JOHN MACLEAN, ...	*Citizen of New Jersey.*
PETER PARKER, ...	" *of Washington.*
WILLIAM T. SHERMAN, ...	" "
ASA GRAY, ...	" *of Massachusetts.*
J. D. DANA, ...	" *of Connecticut.*
HENRY COPPÉE, ...	" *of Pennsylvania.*

MEMBERS EX-OFFICIO OF THE INSTITUTION.

Ulysses S. Grant, ...	*President of the United States.*
Henry Wilson, ...	*Vice-President of the United States.*
Hamilton Fish, ...	*Secretary of State.*
B. H. Bristow, ...	*Secretary of the Treasury.*
W. W. Belknap, ...	*Secretary of War.*
George M. Robeson, ...	*Secretary of the Navy.*
J. A. J. Creswell, ...	*Postmaster-General.*
George H. Williams, ...	*Attorney-General.*
Morrison R. Waite, ...	*Chief Justice of the United States.*
M. D. Leggatt, ...	*Commissioner of Patents.*

HONORARY MEMBER.

COLUMBUS DELANO. *The Secretary of the Interior.*

TABLE OF CONTENTS.[1]

PAGE

ARTICLE I. INTRODUCTION. Pp. 16.

Advertisement iii

List of Officers of the Smithsonian Institution ix

ARTICLE II. (No. 240.) PROBLEMS OF ROTARY MOTION PRESENTED BY THE GYROSCOPE, THE PRECESSION OF THE EQUINOXES, AND THE PENDULUM. By Brevet Maj.-General J. G. BARNARD, Colonel of Engineers, U. S. A., A.M., LL.D., Member of National Academy of Sciences. 1871–1873. 4to. pp. 74.

The Precession of the Equinoxes and Nutation as resulting from the Theory of the Gyroscope 1

On the Motions of Freely Suspended and Gyroscopic Pendulums, and on the Pendulum and Gyroscope as exhibiting the Rotation of the Earth . 15

On the Internal Structure of the Earth considered as affecting the Phenomena of Precession and Nutation 33

New Addendum 42

ARTICLE III. (No. 241.) A CONTRIBUTION TO THE HISTORY OF THE FRESH-WATER ALGÆ OF NORTH AMERICA. By HORATIO C. WOOD, Jr., M.D., Professor of Botany, and Clinical Lecturer on Diseases of the Nervous System in the University of Pennsylvania; Physician to the Philadelphia Hospital, etc. October, 1872. 4to. pp. 274. Twenty-one colored plates.

Preface v

Introduction 1

Fresh-Water Algæ of the United States 9

Class Phycochromophyceæ 9

Order Cystiphoræ 10

Family Chroococcaceæ 10

Order Nematogeneæ 15

Family Oscillariaceæ 16

Family Nostochaceæ 23

Family Rivulariaceæ 43

Family Scytonemaceæ 55

Family Sirosiphonaceæ 67

Class Chlorophyllaceæ 77

Order Coccophyceæ 78

Family Palmellaceæ 78

Family Protococlaceæ 85

Family Volvocineæ 98

Order Zygophyceæ 100

Family Desmidiaceæ 100

Family Zygnemaceæ 159

[1] Each memoir is separately paged and indexed.

PAGE

Order Siphophyceæ 174
Family Hydrogastreæ 175
Family Vaucheriaceæ 176
Family Ulvaceæ 182
Family Confervaceæ 186
Family Œdogoniaceæ 188
Family Chroolepideæ 203
Family Chætophoraceæ 205
Class Rhodophyceæ 213
Family Porphyraceæ 214
Family Chantransiaceæ 215
Family Batrachospermaceæ 217
Family Lemaneaccæ 221
Supplement 225
Geographical List of Species 229
Bibliography 235
Index 249
Explanation of the Plates 253

ARTICLE IV. (No. 262.) An Investigation of the Orbit of Uranus, with General Tables of its Motion. By Simon Newcomb, Professor of Mathematics, United States Navy. October, 1873. 4to. pp. 296.

Introduction 1

Chapter I. Method of Determining the Perturbations of the Longitude, Radius Vector, and Latitude of a Planet by direct Integration.

Notation and general differential formulæ 6
Formation of the required derivatives of the perturbative function . 10
Correction of these derivatives for terms of the second order . . 12
Integration formulæ for perturbations of radius vector . . . 13
Development of functions of rectangular co-ordinates . . . 14
Integration of perturbations of radius vector 17
Formulæ for perturbations of longitude to terms of the second order . 22
Motion of the orbital planes 24
Perturbations of the second order depending on the motion of the orbital planes 25
Reduction of the longitude to the ecliptic 27
Expressions for the latitude 29

Chapter II. Application of the Preceding Method to the Computation of the Perturbations of Uranus by Saturn.

Data of computation 31
Numerical expressions for R and its derivatives 34
Perturbations of radius vector 44
Perturbations of longitude 49
Perturbations of latitude 51

Chapter III. Perturbations of Uranus produced by Neptune and Jupiter

Adopted elements of Neptune 53
Development of R and its derivatives for the action of Neptune . 54
The term of long period between Neptune and Uranus . . . 55
Perturbations of the longitude produced by Neptune . . . 58
Perturbations of the radius vector produced by Neptune . . 60
Perturbations of the latitude produced by Neptune . . . 61
Perturbations produced by Jupiter 62

PAGE

CHAPTER IV. Terms of the Second Order due to the Action of Saturn.
Preliminary investigation of the orbit of Saturn . . . 65
Perturbations of Saturn and Uranus 68
Formation of the expressions for the terms of the second order . 69
Perturbations depending on the square of the mass of Saturn . . 76
Perturbations depending on the product of the masses of Jupiter and Saturn 77

CHAPTER V. Collection and Transformation of the preceding Perturbations of Uranus.
Terms independent of the position of the disturbing planet . . 79
Secular variations 80
Auxiliary expressions on which the perturbations depend . . 81
Reduced expressions for the latitude of Uranus 93
Positions of Uranus resulting from the preceding theory . . 98
Elements III of Uranus 99

CHAPTER VI. Reduction of the Observations of Uranus, and their Comparison with the preceding Theory.
Reduction of the ancient observations 106
Their comparison with the provisional theory 110
Discussion of the modern observations 111
Reduction of the Results to a uniform system 111
Adopted positions of fundamental stars 113
Discussion of corrections to reduce the different observations to a homogeneous system 115
Table of these corrections 120
Results of the observations from 1781 to 1830 122
Observations from 1830 to 1872 126
Table to convert errors of right ascension and declination of Uranus into errors of longitude and latitude 127
Tabular summary of results of observations, 1830 to 1872 . . 131
Corrections to be applied to the positions of Uranus in the Berlin Jahrbuch and the Nautical Almanac to reduce them to positions from the provisional theory 151

CHAPTER VII. Formation and Solution of the Equations of Condition Resulting from the preceding Comparisons.
Expressions of the observed corrections to the longitudes of the provisional theory in terms of the corrections to the heliocentric co-ordinates 158
Expressions of the same quantities in terms of the corrections to the elements of Uranus and the mass of Neptune 161
Table to express errors of heliocentric co-ordinates as errors of elements 162
Discussions and solutions of the equations thus formed . . . 165
Concluded corrections to the elements of longitude . . . 173
Corrections to the inclination and node of Uranus . . . 173

CHAPTER VIII. Completion and Arrangement of the Theory to fit it for Permanent Use.
Correction of the coefficients of the long inequality between Uranus and Neptune for the terms of the second order 178
Concluded elements, or elements IV of Uranus 181
Long-period and secular perturbations of the elements . . . 182
Table of these perturbations from A.D. 1000 until A D. 2200 . . 184
Mean elements of Uranus 184
Expressions for the concluded theory of Uranus 185

PAGE

CHAPTER IX. General Tables of Uranus.

Enumeration of the quantities contained in the several tables . 190
Precepts for the use of the tables 195
Examples of the use of the tables 198
Tables of Uranus 206
Subsidiary tables 279

SMITHSONIAN CONTRIBUTIONS TO KNOWLEDGE.
240

PROBLEMS OF ROTARY MOTION

PRESENTED BY

THE GYROSCOPE,

THE PRECESSION OF THE EQUINOXES,

AND

THE PENDULUM.

BY

BREVET MAJ.-GEN. J. G. BARNARD,
COLONEL OF ENGINEERS U. S. A., A.M., LL.D., MEMBER OF NATIONAL ACADEMY OF SCIENCES.

[ACCEPTED FOR PUBLICATION, OCTOBER, 1871.]

ADVERTISEMENT.

THE three following papers were read at intervals before the National Academy of Sciences, and subsequently presented to the Smithsonian Institution for publication.

JOSEPH HENRY,
Secretary Smithsonian Institution.

THE PRECESSION OF THE EQUINOXES AND NUTATION AS RESULTING FROM THE THEORY OF THE GYROSCOPE.

In a paper published in the American Journal of Science, in 1857, and in Barnard's American Journal of Education[1] [No. 9] of the same year, I remarked:—

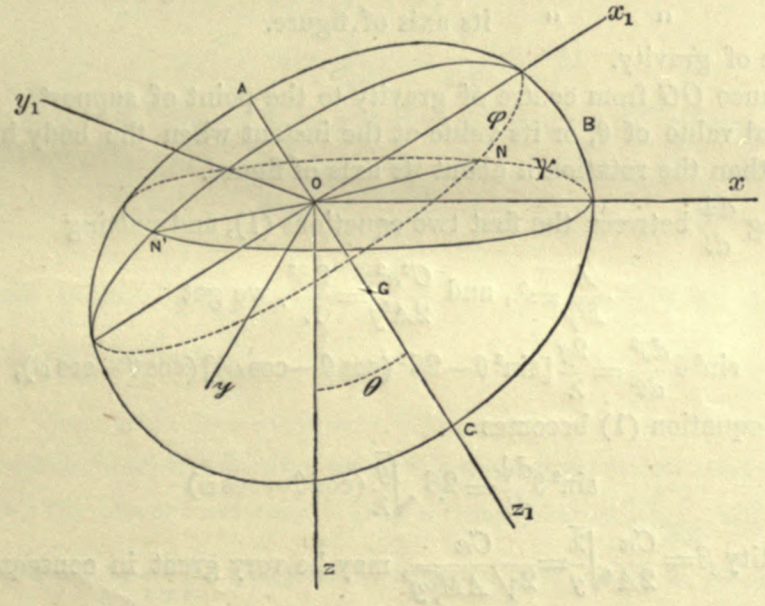
"The analogy between the minute motions of the gyroscope and that grand phenomenon exhibited in the heavens, the 'precession of the equinoxes,' is often remarked. In an ultimate analysis, the phenomena, doubtless, are identical," &c.

It is the object of the present paper to deduce the analytical expressions of this phenomenon directly from the theory of the gyroscope.

A brief summary of the processes used and results arrived at in the paper referred to is necessary as a preliminary.

Let A, B, C, D (Fig. 1) be a solid body of any shape, retained by the fixed point O (within or without its mass). Ox, Oy, and Oz are the three co-ordinate axes,

Fig. 1.



fixed in space, to which the motion of the body is referred. Ox_1, Oy_1, Oz_1 are the three *principal axes* belonging to the point O, and which, of course, partake of the

[1] "The Phenomena of the Gyroscope Analytically Examined."

body's motion. The position of the body at any instant of time is determined by those of the moving axes.

For the purpose of determining the positions of the axes Ox_1, Oy_1, and Oz_1, with reference to the (fixed in space) axes Ox, Oy, Oz, three auxiliary angles are used.

If we suppose the moving plane of $x_1\,y_1$, at the instant considered, to intersect the fixed plane of xy in the line NN' and call the angle $xON=\psi$, and the angle between the planes xy and $x_1\,y_1$ (or the angle zOz_1)$=\theta$, and the angle $NOx_1=\phi$ (in the figure these angles are supposed *acute* at the instant taken), these three angles will determine the positions of the axes Ox_1, Oy_1, Oz_1 (and hence of the body) at any instant, and will themselves be functions of the time: and the rotary velocities about the axes of x_1, y_1, and z_1, may be expressed in terms of them and of their differential coefficients.

When a body is a solid of revolution, revolving with an angular velocity n, about its axis of figure, and acted upon by the accelerating force of gravity (the fixed point O being in the axis of figure), the general equations of rotary motion (by processes fully developed in the paper referred to)[1] take the form

$$1.\qquad \begin{cases} \sin^2\theta\dfrac{d\psi}{dt}=\dfrac{Cn}{A}(\cos\theta-\cos\omega) \\ \sin^2\theta\dfrac{d\psi^2}{dt^2}+\dfrac{d\theta^2}{dt^2}=\dfrac{2Mg\gamma}{A}(\cos\theta-\cos\omega) \\ d\phi=ndt+\cos\theta d\psi \end{cases}$$

In which

M is the mass of the body.

A its moment of inertia about an equatorial axis through O.

C " " " " its axis of figure.

g the force of gravity.

γ the distance OG from centre of gravity to the point of support.

ω the initial value of θ, or its value at the instant when the body has no other motion than the rotation n about its axis of figure.

Eliminating $\dfrac{d\psi}{dt}$ between the first two equations (1), and putting

$$2.\qquad \frac{A}{M\gamma}=\lambda, \text{ and } \frac{C^2n^2}{2A^2g}=\frac{2\beta^2}{\lambda}, \text{ we get}$$

$$3.\qquad \sin^2\theta\frac{d\theta^2}{dt^2}=\frac{2g}{\lambda}[\sin^2\theta-2\beta^2(\cos\theta-\cos\omega)](\cos\theta-\cos\omega),$$

and the first equation (1) becomes

$$4.\qquad \sin^2\theta\frac{d\psi}{dt}=2\beta\sqrt{\frac{g}{\lambda}}(\cos\theta-\cos\omega)$$

The quantity $\beta=\dfrac{Cn}{2A}\sqrt{\dfrac{\lambda}{g}}=\dfrac{Cn}{2\sqrt{AM\gamma g}}$, may be very great in consequence of the rotary velocity, n, being great, or (n being small) in consequence of the ratio

---

[1] The analysis therein used was mostly taken from Poisson as far as equations (of that paper) (9), (10), (11), corresponding to (7), (8), (9) of this; but the subsequent developments were original.

$\frac{C}{2\sqrt{AM\gamma g}}$, being very great. In the phenomena of the gyroscope, the first condition obtains; in the case of the earth, attracted by the sun or moon (n being small), it is easy to show that the alternative condition is fulfilled.[1]

Putting $\frac{d\theta}{dt}$ equal to zero in equation (3) we get $\theta=\omega$ for the *maximum* of θ, and for the minimum, the equation,

$$\cos\theta=-\beta^2+\sqrt{1+2\beta^2\cos\omega+\beta^4}$$

in which, if β is very great, the value of $\cos\theta$ differs but slightly from that of $\cos\omega$. Hence by introducing a new variable u, equal to $\omega-\theta$, and deducing the values of $d\theta$ and (by development) of $\sin^2\theta$ and $\cos\theta$ (neglecting the higher powers of u) and substituting in (3) and (4), they become (omitting, as relatively small, $\cos\omega$ in the factor $\cos\omega+4\beta^2$).

5. $$\sqrt{\frac{g}{\lambda}}dt=\frac{du}{\sqrt{2u\sin\omega-4\beta^2u^2}}$$

6. $$\frac{d\psi}{dt}=2\beta\left(\frac{g}{\lambda}\right)^{\frac{1}{2}}\frac{u}{\sin\omega}$$

Equation (5) gives by integration and putting $\beta\left(\frac{g}{\lambda}\right)^{\frac{1}{2}}=k$.

7. $$u=\frac{1}{2\beta^2}\sin\omega\sin^2kt$$

which substituted in (6) gives

8. $$\frac{d\psi}{dt}=\frac{1}{\beta^2}k\sin^2kt$$

9. $$\psi=\frac{1}{2\beta^2}kt-\frac{1}{4\beta^2}\sin 2kt$$

If we make $\omega=90°$, $\sin\omega=1$, in equation (6), deduce the value of dt, and substitute in (5) we get,

10. $$d\psi=\frac{udu}{\sqrt{\frac{1}{2\beta^2}u-u^2}}$$

the differential equation of the cycloid, generated by a circle of which the diameter is $\frac{1}{2\beta^2}$, and having a chord $\frac{\pi}{2\beta^2}$.

---

[1] For the earth the moments of inertia, A and C, with reference to principal axes through the centre, differ very little. The value of β may therefore be approximately written $\frac{n}{2}\sqrt{\frac{C}{Mg\gamma}}$, and the denominator is to be replaced (17) by $\frac{L}{\sin\theta}$. Substitute the value of L (19) and put, for the sun, $\frac{S}{r^3}=n_1^2$ (25), and the value of β becomes $\frac{n}{2n_1}\sqrt{\frac{C}{3(C-A)\cos\theta}}$, which is very large.

The value as depending on the moon's attraction is (28a), $\frac{n}{2n_2}\sqrt{\frac{C(1+\eta)}{3(C-A)\cos\theta}}$, of the same order of magnitude as before.

If the value of ω is not 90°, the diameter of this circle will be $\frac{1}{2\beta^2}\sin\omega$; but the quantity $\frac{\pi}{2\beta^2}$ then measures an angle of an arc of a small circle having a radius $=\sin\omega$; and the chord of the curve is reduced in the same proportion as its sagitta, and the curve is still a cycloid.

The axis of figure *gyrates* therefore about the vertical through O as if it was attached to the circumference of a small circle of the minute diameter specified, the centre of which circles moves with a uniform horizontal velocity, which velocity is the *mean* rate of gyration; or the motion may be compared to that of a cone having its vertex at O and diameter of base $=\frac{1}{2\beta^2}\sin\omega$ (the axis of figure constituting an element of this cone) rolling upon a fixed conical surface, all the elements of which make, with the vertical, the angle ω; but this imaginary cone *is not fixed in the body* (save in the exceptional case of the moments of inertia A and C being equal). For the rotary velocity of the body is n, while that of the cone is $2\beta\sqrt{\frac{g}{\lambda}}=\frac{Cn}{A}$.

Since the rotary velocities of the body and cone are different, the *instantaneous axis* cannot move along the chord of the cycloid, *nor with uniform velocity*.

The common methods of investigating the Precession of the Equinoxes, founded upon the incipient rate of motion of the instantaneous axis, involve this error, which does not become apparent, simply because the moments of inertia A and C are, for the earth, so nearly equal.

The instantaneous axis will describe a prolate cycloid having the same chord as the common one $\left(\frac{\pi}{2\beta^2}\right)$, and a sagitta $=\frac{1}{2\beta^2}\sin\omega\left(\frac{A}{C}-1\right)$.

The mean rate of gyration is given by the coefficient of t in equation (9); it is $\frac{1}{2\beta}\left(\frac{g}{\lambda}\right)^{\frac{1}{2}}$, or substituting values (2) for β and λ;

11. $$\frac{Mg\gamma}{Cn}$$

Thus far I have supposed that at the origin of time, or at the moment when the accelerating force commenced to act, the body had no other motion than a rotation, n, about its axis of figure. It remains to prove, that if, at this instant, there are *small* (compared to n) velocities about either or both the other principal axes, the rate of gyration will be the same.

The solution will be perfectly general if we suppose at this instant a velocity, m, about the axis of x only, and assume at same moment $\theta=\omega$, $\phi=90°$, we should get, instead of equations (3) and (4), the two following:—

12. $$\sin^2\theta\,\frac{d\theta^2}{dt^2}=\left[\frac{2Mg\gamma}{A}\sin^2\theta-\frac{2Cmn}{A}\sin\omega-\frac{C^2n^2}{A^2}(\cos\theta-\cos\omega)\right.$$
$$\left.-m^2(\cos\theta+\cos\omega)\right](\cos\theta-\cos\omega)$$
$$\sin^2\theta\,\frac{d\psi}{dt}=\frac{Cn}{A}(\cos\theta-\cos\omega)+m\sin\omega$$

Substituting in these $\omega - u$ for θ, rejecting all small quantities of the second order (among which is m^2), and introducing β and λ (see equations 2)

13. $$\frac{\lambda}{g}\frac{du^2}{dt^2}=2u\left(\sin\omega-2\beta\sqrt{\frac{\lambda}{g}}m\right)-4\beta^2u^2$$

14. $$\frac{d\psi}{dt}=2\beta\sqrt{\frac{g}{\lambda}}\frac{u}{\sin\omega}+\frac{m}{\sin\omega}$$

The integral of (13) (using k with its already given value) in (7) is

15. $$u=\frac{1}{2\beta^2}\left(\sin\omega-\frac{2\beta^2}{k}m\right)\sin^2 kt$$

Substitute in (14) and integrate

16. $$\psi=\frac{1}{2\beta^2}kt-\left(\frac{1}{4\beta^2}-\frac{m}{2k\sin\omega}\right)\sin 2kt$$

The coefficient t in (16) is identical with that of equation (9), $=\frac{Mg\gamma}{Cn}$, showing that although the character of the gyratory motion is altered, and the axis of figure, instead of moving on a common cycloid (which forms cusps) and coming periodically to rest, moves along a *prolate* cycloid or even *without undulation*, yet the rate of gyration is unchanged.

If $\omega=90°$ and $m=\frac{Cn}{Mg\gamma}$, u and $\frac{du}{dt}$ become zero for all values of t, and the body gyrates horizontally without nutation.[1]

In all that precedes, the revolving body has been supposed retained by a fixed point in its axis of figure, but *not* at its centre of gravity, while the accelerating force, being gravity itself, acts through that centre.

If, instead, the fixed point by which the body is retained is the centre of gravity, and the accelerating or disturbing forces any other whatever (provided their direction is invariable and their resultant acts through a fixed point of the axis), the

---

[1] This is the case referred to in the preceding paragraph in which the moment of the accelerating force (or couple) is equal to that of (what I have styled in the work before referred to) the "deflecting force," which has for its value the expression $\frac{Cnm}{M\gamma}$.

That this case should arise, a determinate relation between m, n, and g, expressed by the equation $\frac{mn}{g}=\frac{M\gamma}{C}$, is necessary.

When this relation exists, the movement may be represented by the rolling of a conical surface (the locus, in the body, of the instantaneous axis), described about the axis of figure with the angle (approximately) equal to $\frac{Mg\gamma}{Cn^2}$, upon another, all of the elements of which make, with the vertical, the angle $\omega-\frac{Mg\gamma}{Cn^2}$, (when ω is not 90°, the centrifugal as well as the deflecting force affects the relation between m, n, and g).

But no such relation is *essential* to the gyration expressed by (11); and, in the case of the precession of the equinoxes, the supposition of rolling cones is not realized. There are, probably, no two instants of time at which the precessional movements of the axis are identically the same.

equations will be precisely the same, the moments of inertia A and C, referring to principal axes through the centre of gravity, and Mg expressing the intensity of the resultant of the forces, and γ the distance from the centre of gravity of the point through which the resultant acts.

In expression (11) $Mg\gamma$ is the moment of the force with respect to the point O, divided by the sine of the angle (θ) which its direction makes with the axis of figure. Denote that moment by L. Then the expression for the velocity of gyration (11) becomes,

17. $$\frac{L}{Cn \sin \theta}$$

If the body in question, like the earth, is acted upon by forces, the resultant of which does not pass through its centre of gravity, its movements about that centre are precisely the same as if that centre were fixed; in other words, it will gyrate about the line connecting its centre and the origin of the force with a velocity denoted by expression (11). In the case of the earth, however, the direction of the disturbing force and its moment are constantly changing, and I have to assume something not proved in what foregoes, viz., that the *elementary gyration at each moment of time* will be likewise expressed by (11); an assumption *not* (probably) strictly true, since, when the forces are constant in direction and intensity, equation (14) shows (the value of u, equation (15) being substituted) that the gyratory velocity, though its mean is always expressed by (11), varies at each instant unless the value of m has a certain relation to that of k.

Since the integral of these varying elementary displacements shows, under all circumstances of constantly directed force (though these elementary motions of the axis exhibit all possible directions with regard to that of the force), a mean rate of gyration expressed by (11), we may assume that the fact will hold good though the direction and moment of the force change.[1]

In the case of the earth there is probably no instant of time at which it is revolving exactly about its axis of figure; the quantity m has, for it, in all cases, a finite (though exceedingly small) value; neither observation nor (scarcely) analysis can detect the minute diurnal (nearly) nutations which belong to the diurnal cycloidal movement; and hence the presumption that the gyration is *at all instants* perpendicular, or nearly so, to the direction of the force, and hence that even its elementary values vary little from expression (11).[2]

---

[1] Such an assumption is made in all the investigations not, like Laplace's, purely analytical, without always giving the true grounds on which it should be based.

[2] In reality, if the moment L *remains the same* for different values of θ, the elementary *displacement* produced by the gyration is independent of θ, for, though the expression $\frac{L}{Cn \sin \theta}$ varies inversely as $\sin \theta$, yet the radius of the small circle on which the displacement takes place increases in like proportion. Again, that a revolving body should gyrate around a given axis it is not *necessary* that the accelerating force should be always parallel in direction to that axis, but that it should remain in the moving plane through the axis of figure and the given axis. The general equations of rotation would be the same.

Let a, be the equatorial diameter of the earth.

b, be the polar diameter.

ρ, its variable density.

C, its moment of inertia about the polar axis.

A, its moment of inertia about an equatorial one.

$e=\frac{a-b}{a}$, earth's ellipticity.

S, the absolute attractive force of the sun, or its attraction upon a unit of mass at a unit's distance.

r, the mean distance of centres of sun and earth. x, y, z being rectangular co-ordinates of any element of the earth's mass, dm; the origin being the earth's centre, the axis of z the polar one, of x an equatorial one in a plane passing through the sun's centre, of y an equatorial one perpendicular to this plane.

The moment of the sun's attractive force upon the earth is shown in various works on precession (*vide* Mr. Airy's "Figure of the Earth," Encyc. Metropolitana) to be (θ being the angle of earth's axis with line drawn to sun).

18. $$\frac{3S}{r^3}\iiint \rho\,(x^2-z^2)\,dx\,dy\,dz\,\sin\theta\,\cos\theta$$

the integral being taken through the spheroid.

The quantity under the signs of integration may be written

$\rho\,(x^2+y^2)\,dm-\rho\,(y^2+z^2)\,dm$, the integral of the first term of which is C, and of the second A.

Hence the moment of the sun's force (18)

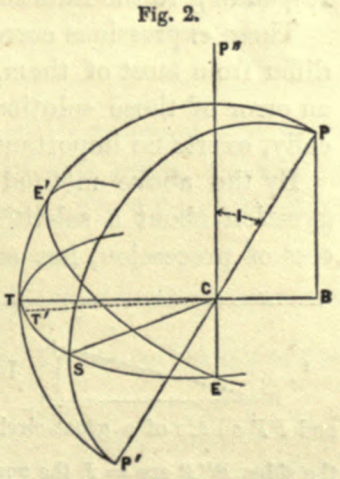
19. $$\frac{3S}{r^3}(C-A)\,\sin\theta\,\cos\theta=L$$

Hence the gyration produced upon the earth by the sun's force about the line of its direction is (17)

20. $$\frac{3S}{r^3n}\frac{C-A}{C}\cos\theta\text{; and in time } dt,$$

21. $$\frac{3S}{r^3n}\frac{C-A}{C}\cos\theta\,dt$$

Fig. 2.



Let EST be a great circle in the plane of the ecliptic, EE' an equatorial one, $PE'TP'$ a great circle through the tropics, PSP' one through the sun in any position, C the centre of the earth, and PCP' its axis. SCP' is the angle θ. If the sun moves in the ecliptic from E (the equinox) towards T with an angular velocity n_1, n_1t will be the value of the arc ES. In the spherical triangle $P'ST$, right-angled at T, we have $\cos P'S$ (or $\cos\theta$)$=\cos P'T\cos TS=\sin TE'\sin SE$.

TE' is the inclination of the equator to the ecliptic; call this I. Then

$$\cos\theta=\sin I\sin n_1t$$

The elementary gyration about the line SC will be therefore, (21)

22. $$\frac{3S}{nr^3}\frac{C-A}{C}\sin I \sin n_1 t\, dt.$$

If this rotation about SC is decomposed into components about the lines TC and EC, they will be

23. $$\frac{3S}{nr^3}\frac{C-A}{C}\sin I \sin^2 n_1 t\, dt.$$

24. $$\frac{3S}{nr^3}\frac{C-A}{C}\sin I \sin n_1 t \cos n_1 t\, dt.$$

The component (23) represents a rotation of the pole about TC, the radius of its motion being PB, or cos I. To obtain the actual value as an arc *of a great circle*, of this minute displacement, it must be multiplied by cos I; and to *refer this to the pole of the ecliptic as angular motion*, it must be divided by sin I.[1] Performing these operations, integrating, and remembering that by Kepler's laws

$\frac{S}{r^3}=\frac{4\pi^2}{T^2}=n_1^2$ (T=number of units of time in one year), we get, for precession,

25. $$\frac{3}{2}\frac{n_1^2}{n}\frac{C-A}{C}\cos I.t-\frac{3}{4}\frac{n_1}{n}\frac{C-A}{C}\cos I \sin 2n_1 t.$$

And for nutation

26. $$\frac{3}{4}\frac{n_1}{n}\frac{C-A}{C}\sin I \cos 2n_1 t.$$

The first term of (25) is the mean solar precession; making $t=\frac{2\pi}{n_1}$, it gives for the annual solar precession

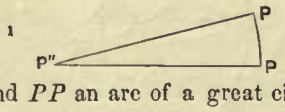
27. $$3\frac{n_1}{n}\pi\frac{C-A}{C}\cos I.$$

Expression (26) is the solar nutation, and the second term of (25) gives the *equation of the equinoxes in longitude*, or the fluctuating term of the precession corresponding to the nutation.

These expressions correspond to those obtained by the ordinary solutions. They differ from most of them, however, in having C in the denominator instead of A, an error of those solutions I have alluded to before, which, however real, analytically, exerts no important influence on the result.

By the above method the precession is the integral of the components of gyration about a solstitial diameter of the ecliptic, which line itself, by the process of precession, has an angular motion equal to that precession, the real effect

---

[1] 

In the spherical triangle $PP''P$ in which P'' is the pole of the ecliptic and PP an arc of a great circle through which the pole P has moved (equal to (23) $\times$ cos I), and the sides $P''P$ are $= I$, the angle $PP''P$ (or the elementary precession)$=\frac{PP}{\sin I}$.

being a revolution about an axis (the pole of the ecliptic) perpendicular to the plane of that angular motion. In other words, if we integrate directly equation (23) and make $t=\frac{2\pi}{n_1}$, and $\frac{S}{r^3}=n_1^2$, we shall get

28. $$3\frac{n_1}{n}\pi\frac{C-A}{C}\sin I;$$

and this will be the total angular motion of the pole P about the solstitial line TC *in one revolution of the sun;* but by this very motion of the pole the equinoxes have moved an angle measured by *this displacement referred to the pole of the ecliptic*—that is, by the angle expressed by (27)—and the solstitial line TC has of course, undergone the same movement, and the next annual gyration will be about the consecutive line $T''C$, and so on; producing a continuous motion of the pole P about the pole of the ecliptic P''.

To obtain the precession due to the moon, it is necessary to substitute in (19) for $\frac{S}{r^3}$, $\frac{M^1}{(r)^3}$, in which M^1 is the attractive force of the moon and (r) its mean distance. But T^1 (time of moon's revolution) is, by Kepler's laws, $\frac{2\pi(r)^{\frac{3}{2}}}{(M^1+E)^{\frac{1}{2}}}=\frac{2\pi(r)^{\frac{3}{2}}}{[(1+\eta)M^1]^{\frac{1}{2}}}=\frac{2\pi}{n_2}$ (calling the mean angular velocity of the moon n_2 and the ratio of earth's mass to that of moon's mass, η) (28*a*); hence $\frac{M^1}{(r)^3}=\frac{n_2^2}{1+\eta}$.

If i is the inclination of the moon's orbit to the equator during any one revolution (regarded as constant for that time), we should obtain for the precession and nutation, *referred to the pole of the moon's orbit*, expressions analogous to (25) and (26).

Although the moon's disturbing effect, as above expressed, is almost exactly double that of the sun, yet the larger divisor n_2, introduced by integration, renders the value of (26) and of the fluctuating term of (25) very small for the moon—say about $\frac{1}{6}$th the corresponding values for the sun. Hence these terms are usually disregarded in the lunar expressions.

The elementary precession due to the moon *about the pole of its own orbit* would be by (25)

29. $$\frac{3}{2}\frac{n_2^2}{n(1+\eta)}\cdot\frac{C-A}{C}\cos i\,dt.$$

From this, by the usual methods, can be deduced the real precession and nutation. But it will be more in harmony with the object of this paper, and indeed more elegant, to reduce the *gyration* produced by the moon directly to precession and nutation.

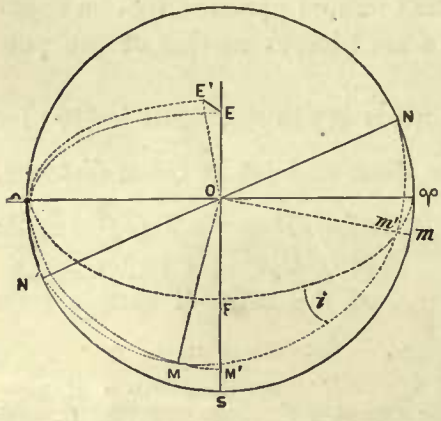
If we substitute for n_1, $\frac{n_2}{1+\eta}$, and sin i for sin I, in (28) we shall get, for the total gyration *about the line of greatest declination*, produced by one revolution of the moon in its orbit, the expression:

30. $$3\frac{n_2}{n(1+\eta)}\pi\frac{C-A}{C}\sin i.$$

In which the line of greatest declination is regarded as stationary during the single revolution, and taking a consecutive position for the next; but it will be in harmony with the fact, and allowable, to regard the line as in continuous motion and the above amount of gyration to be uniformly spread over the time $\frac{2\pi}{n_2}$, of the revolution, producing thus an elementary gyration, in the time dt, of

31. $$\frac{3}{2}\frac{n_2^2}{n(1+\eta)}\frac{C-A}{C}\sin i\, dt.$$

Let $\Upsilon S\backsim$ be a great circle in the plane of the ecliptic; $\Upsilon\backsim$ the line of equinoxes, NON' the line of moon's nodes, $\Upsilon F\backsim$ the equator, and $Nm'MN'$ the moon's orbit crossing the equator at m'. The line of the moon's maximum declination, OM, will be 90° from the line Om'.



The pole E of the earth is supposed to undergo a displacement by gyration about OM represented by EE'; the precession produced will be the angle EOE'; the nutation, the angle $E\backsim E'$.

In the spherical triangle $Nm'\Upsilon$ the angle at N is $= I'$, the inclination of moon's orbit to ecliptic; the angle at Υ is the supplement of I (inclination of the equator) and the angle at m' is i (or the variable inclination of the moon's orbit to the equator), and the side ΥN is $= n_3 t$ (calling the angular velocity of the moon's node n_3); therefore,

32. $$\cos i = \cos I' \cos I + \sin I' \sin I \cos n_3 t$$

and

33. $$\text{tang } m'\Upsilon = \frac{\sin n_3 t}{\sin I \cot I' - \cos I \cos n_3 t}.$$

In the spherical triangle $mm'\Upsilon$

34. $$\text{tang } m\Upsilon = \cos I \text{ tang } m'\Upsilon$$

OS is the line of maximum declination of the sun, or the solstitial diameter of the ecliptic about which the annual gyration produced by the sun is made. As the inclination I' of the moon's orbit is small, the arc MM', drawn through $M\backsim$, is approximately equal to $m\Upsilon$, and the angle $MO\backsim$ differs immaterially from the complement of $m\Upsilon$; hence by (33) and (34)

35. $$\text{tang } MM' = \frac{\cos I \sin n_3 t}{\sin I \cot I' - \cos I \cos n_3 t} = \frac{\cos I \sin I' \sin n_3 t}{\sin I \cos I' - \cos I \sin I' \cos n_3 t}.$$

If the gyration about OM (31) is decomposed into components about OM' and $O\backsim$, we shall have for the first (calling the coefficient of sin $i.dt$, K)

36. $$K \sin i \cos MM' dt,$$

and for the second

37. $$K \sin i \sin MM' dt.$$

To refer the *displacement* expressed by (36) to the pole of the ecliptic O, as angular motion, we must (see note page 8) multiply by cos $M'F$ or (from which it differs but slightly) cos i and divide by sin I; we thus obtain for the elementary *precession*

38. $$\frac{K}{\sin I}\cos MM' \sin i \cos i\, dt,$$

and for nutation

39. $$K \sin MM' \sin i\, dt.$$

The maximum value of the arc MM' is about 11° 56′; the line MO describing during an entire revolution of the moon's nodes an elliptical cone about OS of which the minor semi-diameter (SM') is 5° 8½′ (about), and the semi-major 11° 56′. By conceiving the elementary motion of the pole of the earth (or its gyration) as at each instant about the line MO, as it makes its conical revolution, the undulating nature of that motion, or the "nutation," is easily conceived.

Approximate values of sine and cosine of MM' may be determined from (35); which, substituted with those of i (32) in (38) and (39), will enable us to integrate and obtain very accurate expressions for the lunar precession and nutation.[1] But these expressions, nearly free from errors of approximation, may be more elegantly determined as follows: When the angle of the moon's orbit with the equator is minimum, the angle $i=I-I'$; when maximum $i=I+I'$ (epochs corresponding to $n_3t=0$ and $n_3t=\pi$); the angle MM' is zero, and the corresponding *rates* of precession are by (38)

(*a*) $$K\frac{\sin 2(I-I')}{2 \sin I} \text{ for } n_3t=0.$$

(*b*) $$K\frac{\sin 2(I+I')}{2 \sin I} \text{ for } n_3t=\pi.$$

When $n_3t=\frac{1}{2}\pi$, we have

$$\cos MM'=\frac{\sin I}{\sqrt{1-\cos^2 I \cos^2 I'}}, \quad \sin MM=\frac{\cos I \sin I'}{\sqrt{1-\cos^2 I \cos^2 I'}}$$

$\cos M'F=\cos I \cos^2 I'$, $\cos i=\cos I \cos I'$, $\sin i=\sqrt{1-\cos^2 I \cos^2 I'}$ (the three first being the residuals of exact analytical expressions after omission of quantities of inappreciable magnitude).

Hence, by (38), the rate of precession for $n_3t=\frac{1}{2}\pi$ is

(*c*) $$K \cos I \cos^2 I' \text{ *}$$

Assume the formula for precession to be

$$K(Pt+P' \sin n_3t+P'' \sin 2n_3t);$$

---

* When the moon's orbit intersects the ecliptic in a solstitial line, the elementary precession $K \cos i$ (29) *about its own pole* is reduced, with but slight error, to the same about the ecliptic pole, by simple multiplication by cos I': a result coinciding with the above.

[1] See Additional Notes, p. 51.

the *rates* of precession will be

(a')	$K(P+\ n_3P'+2n_3P'')$	for $n_3t=\ 0$
(b')	$K(P-\ n_3P'+2n_3P'')$	" $n_3t=\ \pi$
(c')	$K(P-2n_3P'')$	" $n_3t=\frac{1}{2}\pi$

Equating (a) to (a'), &c., we deduce

$$P=\tfrac{1}{2}\cos I(\cos^2 I'+\cos 2I')=\cos I\left(1-\frac{3}{2}\sin^2 I'\right)$$

$$P'=-\frac{1}{2n_3}\frac{\sin 2I'\cos 2I}{\sin I}=-\frac{1}{n_3}\cos I\sin 2I'\cot 2I$$

$$P''=-\frac{1}{4n_3}\cos I(\cos I'-\cos 2I')=-\frac{1}{4}n_3\cos I\sin^2 I'.$$

Hence the formula for precession may be written

44. $$\frac{3}{2}\frac{n_2^{\,2}}{n(1+\eta)}\frac{C-A}{C}\cos I\left[\left(1-\frac{3}{2}\sin^2 I'\right)t-\frac{1}{n_3}\sin 2I'\cot 2I\sin n_3t-\frac{1}{4n_3}\sin^2 I'\sin 2n_3t\right].$$

Similarly we would get for the nutation

45. $$\frac{3}{2}\frac{n_2^{\,2}}{n(1+\eta)}\frac{C-A}{C}\cos I\frac{\sin I'}{n_3}\cos n_3t.$$

The ratio of actual lunar precession to what it would be were the moon's orbit in the ecliptic, is therefore expressed by

$$1-\frac{3}{2}\sin^2 I'=0.99 \text{ (very nearly).}^1$$

The third term of (44) indicates a slight periodical variation from the true elliptic motion referred to in the next paragraph. There should be a corresponding term in (45) which may be obtained by the same process, but they are both too minute to enter into computations.

---

[1] It is worthy of remark that the formulæ of Laplace [3100] and [3101] (Bowditch) contain no such coefficient qualifying the mean lunar precession, though one is found in all the more popular solutions; neither do they contain the term (quite minute) in $2n_3t$ of (44), but, on the other hand, contain terms in $2n_2v$ (corresponding to the terms in $2n_1t$ of 25 and 26), which, referred to in the fourth par. (page 9), are generally omitted as inappreciable.

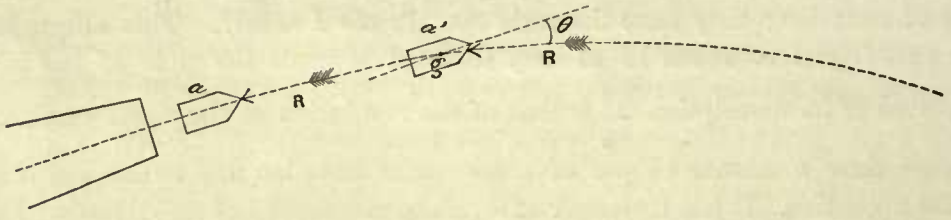
If we multiply the coefficient of sin $n_3 t$ in (44) by sin I, we shall have the value as an arc of a great circle of this fluctuating displacement called *the equation of the equinoxes in longitude*. The coefficient thus modified will represent the minor semi-diameter, and that of the nutation proper (45), the major semi-diameter of the ellipse of nutation; they have the ratio cos $2I$: cos I nearly. This ellipse has its major axis (equal to about 18″ of arc) directed towards the pole of the ecliptic. The period of its description $\frac{2\pi}{n_3}$ is that of the revolution of the moon's nodes. At the same time a minute ellipse of semi-annual nutation due to the sun is superimposed upon this. It has its longer axis (about one second of arc) likewise directed to the pole of the ecliptic. The smaller axis is to the major as cos I: 1.

It is easy to show that the precession caused by the sun and moon is equal (with slight difference due to the ratio we have just been considering) to what it would be if those bodies were uniformly distributed in solid rings over circles (in the plane of the ecliptic) about the centre of the earth, having radii equal to their mean distances. In this case, unless there was a particular relation between the couple producing the initial rotation of the earth and that arising from the attraction of the two rings, there would be an extremely minute nutation, of which the period would be (see equation 9) $\frac{2\pi}{2k} = \frac{2\pi}{n}\frac{A}{C}$; which is almost identical with the siderial day $\left(\frac{2\pi}{n}\right)$, the ratio being $\frac{A}{C}$: 1. If we suppose the primitive rotation of the earth to be that alone, about its axis of figure n, then the nutation will exhibit the common cycloidal motion of equation (10); but its total amount would be but about $\frac{1}{5}$ second of arc.

An explanation of the deviation of rifled projectiles will be found in what is said (pages 8 and 9) in reference to the conversion of gyration about a shifting axis into precession about an axis perpendicular to the plane of motion of the first. Elongated rifled projectiles, while they maintain almost unaltered their "angle of elevation," are found to deviate with great uniformity from the vertical plane of projection, in a direction corresponding to the twist of the gun, while spherical projectiles (fired from rifled guns) having precisely the same rotary motion, do not so deviate; showing that the cause of the phenomenon is something else than the direct action of friction or pressure of the air.

In the remarks made in the paragraphs just referred to, it is explained how the consecutive small annual gyrations about the line from the centre of the earth to the sun (in the tropics) become, in their integral, the movement which we call precession, about an axis perpendicular to the plane of motion of that line, inasmuch as each small primary gyration causes a corresponding shifting of the line about which it takes place. Something very similar occurs to produce the deviation of elongated projectiles. In issuing from the gun, the resultant of the atmospheric resistance (denoted by the arrow R) coincides with the axis of the projectile, and it has no other effect than to retard the motion of translation; but the action of gravity causes the trajectory to curve downwards, and the direction of the atmospheric resistance becomes oblique to the axis of the projectile (at a'), and (in

almost all forms of projectiles) passes *above*, not through, the centre of gravity g. We have then the essential conditions of gyration, viz., a solid of revolution revolving rapidly about its axis, and a dynamic "couple" (*i. e.*, the inertia of the



projectile's motion of translation acting through its centre of gravity, and the resistance of the air acting through a point of the axis more or less distant from g) tending to turn the projectile upwards about a horizontal (or "equatorial") axis through g, and there is in fact, at each instant, an elementary gyration about a line through g, parallel to R (the atmospheric resistance). If this line retained an invariable direction, the integral effect of these elementary gyrations would be to revolve *down* the axis of the projectile, and we should ultimately find it assuming horizontal and even sub-horizontal directions. But such cannot be the case; the direction of the axis is no sooner deviated, laterally, from its original direction, than a (nearly) corresponding change takes place in the direction of the resistance R (since from the elongated form of the projectile, the direction of its motion follows pretty nearly that of its axis) and in that of the line (parallel to R) about which gyration takes place. The integral of such a series of elementary gyrations, accompanied by a *corresponding* horizontal angular motion of the line about which they take place, is angular motion about a line perpendicular to the plane in which that line shifts direction, that is, about a vertical. Hence the vertical direction (or "elevation") of the axis of the projectile remains constant, or nearly so, while its horizontal direction undergoes a progressive angular precession (if I may so term it), and the deviation of rifled projectiles is thus seen to have analogy with the precession of the equinoxes.[1]

In what precedes I do not profess to throw new light on a subject so thoroughly studied as the Precession of the Equinoxes; my object has been rather to make evident the analogy that exists between "the minute motions of the gyroscope and that grand phenomenon exhibited in the heavens," and to show how a common analysis applies to both.

---

[1] It is quite probable that there are other causes of deviation, the friction of the air being (in case of long ranges) one. Experimental facts are needed for a full discussion of this subject.

ON THE

MOTIONS OF FREELY SUSPENDED AND GYROSCOPIC PENDULUMS, AND ON THE PENDULUM AND GYROSCOPE AS EXHIBITING THE ROTATION OF THE EARTH.

Let the point of suspension of the pendulum be taken as the origin of rectangular co-ordinates, the axis of z vertical (downwards), and those of x and y in the plane of the horizon, the former directed to the east, the latter to the north.

The forces which act on the pendulum (considered as concentrated in its centre of oscillation) are

1st. Gravity, the resultant of the earth's attraction, and of the centrifugal force of its rotation.

2d. The tension of the pendulum cord.

3d. The force of inertia, the components of which are represented by the differential coefficient $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$; and

4th. The disturbing forces arising from the earth's rotation.

5th. The resistance of the air.

If we represent the length of the string by l, its tension by N, and the force of gravity by g, and neglect the forces named in the 4th and 5th categories, we shall have the three equations:

$$(1) \qquad \begin{aligned} \frac{d^2x}{dt^2} &= -\frac{Nx}{l} \\ \frac{d^2y}{dt^2} &= -\frac{Ny}{l} \\ \frac{d^2z}{dt^2} &= -\frac{Nz}{l} + g. \end{aligned}$$

The forces due to the earth's rotation which disturb the *relative* motions of a projectile, or any material particle moving near the earth's surface, have been expressed by Poisson (Journal de l'Ecole Polytechnique Cahier, 26) as follows:—[1]

---

[1] These expressions are perfectly general and applicable to *all* problems which involve relative motions near the earth's surface, whether of solids or fluids. They (or their equivalent) appear in the work of Laplace (Mec. Cel., Vol. IV.), as well as of Poisson (from whom I have quoted them), in the investigations of the motions of projectiles. It is somewhat extraordinary that both of these great analysts should have failed to perceive their remarkable application in the motions of the gyroscope and pendulum, to the exhibition of the earth's rotation, although the latter has seemed to desire such an exhibition in the sentence: "Quoique la rotation de la terre soit maintenant établie

$$X = 2n\left(\frac{dy}{dt}\sin\lambda + \frac{dz}{dt}\cos\lambda\right)$$

(2) $$Y = -2n\,\frac{dx}{dt}\sin\lambda$$

$$Z = -2n\,\frac{dx}{dt}\cos\lambda.$$

In which λ is the latitude and n the angular velocity of rotation of the earth.

These analytical expressions make their appearance in the transformations of the equations of motion, near the earth's surface, expressed in co-ordinates referring to *fixed* axes, into others referring to the moving axes which are used in this analysis. But these forces can be obtained and their origin better understood by the following considerations.

The centrifugal force of a material point at rest on the earth's surface at the given latitude will be $\frac{n^2 r^2 \cos^2\lambda}{r\cos\lambda}$ (r being the earth's radius). If it has a small relative velocity $\frac{dx}{dt}$ to the east, the centrifugal force will become $\frac{\left(n\,r\cos\lambda + \frac{dx}{dt}\right)^2}{r\cos\lambda}$ Subtracting the former expression from the latter (omitting $\frac{dx^2}{dt^2}$) we get for the centrifugal force arising from the relative velocity $\frac{dx}{dt}$ the expression $2n\frac{dx}{dt}$. The component of this in the direction of the axis of y will be $-2n\sin\lambda\frac{dx}{dt}$, which corresponds to the value of Y (equations 2) of Poisson, and is to be added to the second member of the second of equations (1).

The component of the force just calculated in the direction of the axis of Z is

$$-2n\cos\lambda\frac{dx}{dt}.$$

This force corresponding to Poisson's value of Z is to be added to the second member of the third of equations (1).

A body moving on a meridian of the earth's surface from south to north will have the moment of its quantity of motion, with reference to the earth's axis, diminished; in virtue of which it will press with a certain force towards the east

---

avec toute la certitude que les sciences physiques comportent, cependant une *preuve directe* de ce phénomène doit interesser les géomètres et les astronomes." The former, in making a partial application to the pendulum, of his investigations, absorbed, apparently, in the single object of proving that the accuracy of the instrument as a measure of time was not affected, has inadvertently assumed that the disturbing force normal to the plane of oscillation is "trop petite pour écarter sensiblement le pendule de son plan et avoir aucune influence appreciable sur son mouvement." (Journal de l'Ecole Poly. Cahier, 26, p. 24.) It is true, indeed, that the force he mentions, even if permitted free action, will have but an inappreciable influence upon the time, and none whatever when, as in the chronometer, the plane is constrained to fixedness; but the effect is *cumulative* in changing the azimuth of the freely suspended pendulum.

These same disturbing forces, introduced along with the attractions of the sun, moon, and earth, into the general equations of equilibrium of fluids, produce in a very simple manner the differential equations for the tidal motions. (*Vide* American Journal of Science, 1860.)

and the moment of the force thereby developed is equal to $\frac{d}{dt}$ (the moment of its quantity of motion). This moment for the pendulum in any latitude λ, is $nr^2 \cos^2 \lambda$, of which the differential coefficient, taken with reference to λ as a function of t, is $-2nr^2 \sin \lambda \cos \lambda \frac{d\lambda}{dt}$, which is the moment of the force required. Dividing by the radius of rotation, $r \cos \lambda$, we have $+2nr \sin \lambda \frac{d\lambda}{dt}$ for the expression of the force which (acting positively in the direction of the axis of x) is to be taken with the plus sign.

In the system of rectangular co-ordinates which I am using $\frac{dy}{dt}$ corresponds to the velocity expressed by $r \frac{d\lambda}{dt}$; and substituting it therefor, we have for a disturbing force in the direction of the axis of x the expression

$$2n \sin \lambda \frac{dy}{dt}$$

In almost precisely the same way it may be shown that a body falling towards the centre of the earth with a velocity $\frac{dz}{dt}$ will have the moment of its quantity of motion diminished by $2nr \cos^2 \lambda \frac{dz}{dt}$, giving rise to the force

$$2n \cos \lambda \frac{dz}{dt}$$

The sum of these two expressions constitutes the disturbing force X of Poisson, and is to be added to the second member of the first of equations (1), and these equations become[1]

$$(3) \qquad \begin{aligned} \frac{d^2x}{dt^2} + \frac{Nx}{l} &= 2n \sin \lambda \frac{dy}{dt} + 2n \cos \lambda \frac{dz}{dt} \\ \frac{d^2y}{dt^2} + \frac{Ny}{l} &= -2n \sin \lambda \frac{dx}{dt} \\ \frac{d^2z}{dt^2} + \frac{Nz}{l} &= g - 2n \cos \lambda \frac{dx}{dt} \end{aligned}$$

---

[1] There are really other disturbing forces (comparatively slight indeed) than the X Y and Z of Poisson (equation 2), as appears from the following considerations :—

Draw a line through the origin of co-ordinates parallel to the axis of the earth, and project the moving body on the plane of $y\,z$. The distance of the projection from the line will be $y \sin \lambda + z \cos \lambda$, the distance of the body from the plane of $y\,z$ being x: hence there will be a centrifugal force relatively to this line, due to the earth's rotation, tending to increase the ordinates $x\,y\,z$ by its components

$$\begin{aligned} &n^2 x \\ &n^2 \sin \lambda \,(y \sin \lambda + z \cos \lambda) \\ &n^2 \cos \lambda \,(y \sin \lambda + z \cos \lambda) \end{aligned}$$

With these expressions added, respectively, to the second members of equations (3), they correspond to those found in Carmichael (Calcul. of Operations), who quotes from Galbraith and Houghton (Proc. R Irish Acad., 1851). They express forces of the second order in minuteness, compared with those expressed by equations (2), and, insensible in their effects, are neglected in all discussions. They are noticed here only to recognize their existence and to show their origin.

3 January, 1872.

Since $x^2+y^2+z^2=l^2$, we have $x\,dx+y\,dy+z\,dz=O$. Hence multiplying equations (3), respectively, by dx, dy, and dz, and adding, we have

$$\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{dt^2}=g\,dz$$

$$(3)_2 \qquad \frac{dx^2+dy^2+dz^2}{dt^2}=2gz+c$$

This expression is independent of n, and the velocity at any point of the path depends, in the same way as does that of a pendulum vibrating over a motionless earth, upon the height of fall. The plane of vibration of the chronometer pendulum is maintained in a fixed relative position, thereby differing from a "freely suspended" pendulum. It will be seen hereafter, in treating of the gyroscope pendulum, that the forces which maintain this relative fixedness are equivalent to a force varying directly as the angular velocity, applied at the centre of gravity, normally to the path. Such a force will have no influence upon the velocity. Hence the time of vibration of the chronometer pendulum is not affected by the earth's rotation, nor by the azimuth angle of the plane of vibration.[1]

Multiplying the first of equations (3) by y, and the second by x, and adding, we get:—

$$y\frac{d^2x}{dt^2}-x\frac{d^2y}{dt^2}=2n\sin\lambda\left(y\frac{dy}{dt}+x\frac{dx}{dt}\right)+2n\cos\lambda\,y\frac{dz}{dt}$$

Integrating:—

$$(4) \qquad y\frac{dx}{dt}-x\frac{dy}{dt}=n\sin\lambda\,(x^2+y^2)+C+2n\cos\lambda\int yaz$$

The above (4) expresses that the moment of the quantity of motion about the axis of z is equal to a constant C (depending upon any arbitrarily given initial value) increased by what is due to the constant angular motion $n\sin\lambda$, and by the area $2\int y\,dz$ (in the case of ordinary plane vibration this is the projection on

---

[1] This conclusion is not invalidated by the introduction of the disturbing forces of the order n^2 referred to in note to p. 17, for, since the arc of vibration of the chronometer pendulum is exceedingly small, z may be considered as equal to l, the pendulum's length, and y as very minute. Those forces will thence be

$$n^2x$$
$$\tfrac{1}{2}\,n^2l\sin 2\lambda$$
$$n^2\,l\cos^2\lambda$$

The third of these is an *increment to gravity*, and the first tends to prolong vibrations in the prime vertical. The second is null in its effects, since, being always positive, it retards the vibration in one direction as much as it accelerates it in the other. But they are all *inappreciably* minute, the last being, for the seconds pendulum, an increment to the force of gravity at the equator of about $\frac{1}{1.800.000.000}$, decreasing the time of vibration by about $\frac{1}{3.600.000.000}$. The first has the contrary tendency to increase the time of vibrations if made in a prime vertical (or any other plane than a meridian), but its effect is equally inappreciable even when (as in the prime vertical) it is a maximum.

the plane of yz of the circular segment included between the arc and chord of vibration), multiplied by $n \cos \lambda$. This multiplier, which is the component of the earth's rotation about a diameter of the earth normal to that passing through the locality, indicates that the term $2n \cos \lambda \int y\, dz$ expresses a disturbance produced by this complementary component. As this term for vibratory motion is small and periodic, passing through nearly equal positive and negative values in the course of a double vibration, it follows that $n \sin \lambda$ expresses the mean increment of angumortion, or, in other words, that, to the plane or spherical vibrations exhibited by the pendulum over a motionless earth, there is, superadded, in consequence of this rotation, a uniform azimuthal motion measured by the earth's rotating velocity multiplied by the sine of the latitude. This is the material fact or peculiar feature of the freely suspended pendulum, and we see that it is exhibited by equation (4) *generally* for all *ordinary* vibrations, whether plane or spherical. We shall see hereafter, however, that the disturbing term of equation (4) $2n \cos \lambda \int y\, dz$ expresses a tendency to a like motion about the complementary axis, and that, on the supposition of an infinite velocity, this tendency may be realized, and the plane of motion, by the joint effect of the two components, turn around a parallel to the earth's axis, with an angular velocity equal and contrary in direction to n.

In the case of *very small* deviations from the vertical, the equations (3) may be solved as follows: The variations of z then become of the second order of minuteness compared with those of x and y, and omitting them we have between x and y and their differentials the relations

$$(4)_2 \qquad \begin{aligned} \frac{d^2x}{dt^2} + \frac{Nx}{l} - 2n \sin \lambda \frac{dy}{dt} &= 0 \\ \frac{d^2y}{dt^2} + \frac{Ny}{l} + 2n \sin \lambda \frac{dx}{dt} &= 0 \end{aligned}$$

the integrals of which are (Gregory Examp. p. 390),

$$(5) \qquad \begin{aligned} x &= +\Sigma (D \cos \beta t + E \sin \beta t) \\ y &= -\Sigma (D \sin \beta t - E \cos \beta t) \end{aligned}$$

in which to β is given both the values obtained from the quadratic equation

$$a_0 - a_2 \beta^2 = a_1 \beta \text{ *}$$

in which $a_0 = \frac{g}{l}$, $a_1 = -2n \sin \lambda$, $a_2 = 1$; hence solving the quadratic

$$\beta = -\frac{1}{2} \frac{a_1}{a_2} \pm \sqrt{a_0 + \frac{1}{4} \left(\frac{a_1}{a_2}\right)^2}$$

Substituting the values of a_0, &c., and omitting the second term under the radical as inappreciably small, we have

---

* This equation, $\frac{g}{l} - \beta^2 = 2n \sin \lambda\, \beta$, can be got by substituting the integrals $x = C \cos (\beta t - \varepsilon)$, $y = C \sin (\beta t - \varepsilon)$ in the given equations.

$$\beta = n \sin \lambda \pm \sqrt{\frac{g}{l}}$$

Representing these two values by β_1 and β_2, equations (5) may be put in the form (by writing for D_1 and E_1, $C_1 \cos \varepsilon_1$ and $C_1 \sin \varepsilon_1$, &c., and reducing)

$$x = C_1 \cos(\beta_1 t - \varepsilon_1) + C_2 \cos(\beta_2 t - \varepsilon_2)$$
$$y = -C_1 \sin(\beta_1 t - \varepsilon_1) - C_2 \sin(\beta_2 t - \varepsilon_2)$$

Assuming for $t=0$, $x=0$ and $\frac{dy}{dt}=0$, it will give $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}\pi$, and the above become

$$x = C_1 \sin \beta_1 t + C_2 \sin \beta_2 t$$
$$y = C_1 \cos \beta_1 t + C_2 \cos \beta_2 t$$

Instead of the arbitrary constants C_1 and C_2 we may write $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, at the same time substituting the values of β_1 and β_2 and developing; by which the preceding equations become (putting $n \sin \lambda = n'$)

(6)
$$x = A \cos \sqrt{\frac{g}{l}}\, t \sin n' t + B \sin \sqrt{\frac{g}{l}}\, t \cos n' t$$
$$y = \mathrm{A} \cos \sqrt{\frac{g}{l}}\, t \cos n' t - B \sin \sqrt{\frac{g}{l}}\, t \sin n' t$$

If we transfer the co-ordinates now referring to (relatively) *fixed* axes, to others *moving* with the relative angular velocity n', that is, if we transfer to axes making at any instant the angle $n \sin \lambda\, t$ with the fixed ones, the new co-ordinates will have the values

$$x' = x \cos n't - y \sin n't$$
$$y' = x \sin n't + y \cos n't$$

or, substituting values of x and y,

$$x' = B \sin \sqrt{\frac{g}{l}}\, t$$

$$y' = A \cos \sqrt{\frac{g}{l}}\, t$$

From which we may obtain

$$A^2 x'^2 + B^2 y'^2 = A^2 B^2$$

which is the equation of an ellipse, having A and B for semi-transverse and semi-conjugate axes. If $B=0$ the ellipse becomes a right line, hence the earth's rotation causes an azimuthal motion of this line, or of the axes of the ellipse if the motion is elliptical, equal to the component of that rotation about the local axis and in the reverse direction.

The motions of the "gyroscope pendulum," which is but the ordinary gyroscope with an exceedingly long arm (or distance γ, of my analysis, from the point of support in the axis to the centre of gravity), are indicated by equations precisely similar to the above, deduced from an identical analysis; always assuming, as in the solutions just given, that the arcs of vibration are small, so that *vertical* motions

may be disregarded. To prove this, I refer to my expression (h) and the context, in my analysis of the "Gyroscope."[1]

(α) $$-\frac{Cn}{\gamma M} v_s$$

for the deflecting force, as I call it (a force due to the rotation of the disk with angular velocity n, and acting, at the centre of gravity, normally to the plane of angular motion of the disk-axis, or of the arm of the gyroscope pendulum), in which M is the mass, C its moment of inertia about the disk-axis, γ the distance from its centre of gravity to point of suspension, and v_s the angular velocity.

Disregarding the vertical motions represented by $\frac{dz}{dt}$ on account of the smallness of the arcs, $\frac{1}{l}\frac{dx}{dt}$ and $\frac{1}{l}\frac{dy}{dt}$ (I substitute l for the γ mentioned above) would represent very nearly the components of angular velocity of the centre of gravity. Substituting these for v_s we shall get the components of the "deflecting force," and the equations of motion will be,

$$\frac{d^2x}{dt}+\frac{Nx}{l}=-\frac{Cn}{l^2M}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}+\frac{Ny}{l}=+\frac{Cn}{l^2M}\frac{dx}{dt}$$

$$\frac{d^2z}{dt}+\frac{Nz}{l}=+g$$

These equations are identical in all but the value of the coefficients with (3), when transformed to $(3)_2$, under the same license. Of course the motions of the gyroscope pendulum would have the same solutions, the mean azimuthal motion of the nodes of its orbit being expressed by half the coefficient of $\frac{dy}{dt}$, $-\frac{Cn}{2l^2M}$, or, since the moment of inertia A, of the gyroscope, with reference to a principal axis through the point of support, is (l being supposed to be very large compared to the dimensions of the disk) very nearly l^2M, the mean azimuthal motion is more simply expressed by $\frac{Cn}{2A}$.

This may be more generally proved as follows: The first of the general differential equations (equations 4 of my analysis) of gyroscopic motion is

$$\sin^2\theta\,\frac{d\psi}{dt}=\frac{Cn}{A}(\cos\theta-c)$$

in which θ, counted from the inferior vertical, is the variable inclination, and ψ the azimuth angle of the disk-axis or pendulum arm, and c a constant depending on initial values of θ and $\frac{d\psi}{dt}$.

[1] See American Journal of Science, 1857, and Barnard's American Journal of Education, 1857.

Develop $\cos\theta = -(1-\sin^2\theta)^{\frac{1}{2}}$ and we get

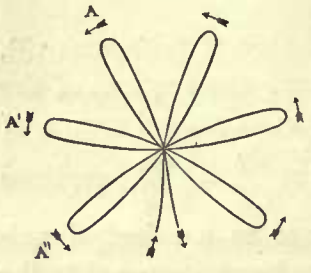
$$\frac{d\psi}{dt} = -\frac{Cn}{2A}\left(1+2\frac{1-c}{\sin^2\theta}-\tfrac{1}{4}\sin^2\theta-\tfrac{1}{8}\sin^4\theta-\&c.\right)$$

For ordinary ranges of pendulum vibration the terms involving positive powers of $\sin\theta$ (which express an excess of nodal motion for large excursions) may be omitted, and we have

$$\frac{d\psi}{dt} = -\frac{Cn}{2A}+\frac{Cn}{A}\frac{1-c}{\sin^2\theta}$$

Thus we see that for small vibrations, whether spherical or plane, the azimuthal motion is made up of a uniform progression of the nodes $\frac{Cn}{2A}$, and a fluctuating term which represents the angular velocity *in the orbit*. Indeed we have, in the second term, the motions of the spherical pendulum.

If we suppose the pendulum to have been propelled from a state of rest in the vertical, $\frac{d\psi}{dt}$ must have a finite value when θ is indefinitely small, and c must hence be unity. Hence we see that at the very outset the initial value $\frac{Cn}{2A}$ must be attributed to $\frac{d\psi}{dt}$, and that the pendulum reacquires it at every return excursion, that is, whenever θ diminishes indefinitely. Hence the pendulum continues to pass through the vertical at every return. The horizontal projection of the curve would be a series of loops radiating from a common centre. For each complete vibration the integral of $\frac{Cn}{2A}dt$ would represent the entire angular motion of the nodal axis (much exaggerated in the diagram) from A to A', &c., and the integral of the remaining terms should be 2π. These loops are in fact but the path a pencil attached to a common pendulum would trace upon a paper beneath, turning with uniform angular velocity about the projection of the point of suspension.



Though the numerator of the fraction $\frac{Cn(1-c)}{A\sin^2 0}$ becomes zero for the case just considered, it is evident that, at the moment of passing through the vertical, the limiting value must be considered infinity, and that the integral through the infinitely short time of passage must be π; for the azimuthal position undergoes, at that instant, an increment (or decrement) of a semi-circumference. There is an identical case in the spherical pendulum. Regarding plane as the final limit of narrowing spherical vibrations, it is evident that the azimuthal velocity of passage by the vertical becomes very great and has its limit infinity when they pass *through* the vertical.

This expression $\frac{Cn}{2A}$ (equal to $\beta\sqrt{\frac{g}{\lambda}}$ nearly) is a very different thing from the "mean precession" of the gyroscope, $\frac{1}{2\beta}\sqrt{\frac{g}{\lambda}}$, given in my analysis. The latter is

the mean azimuthal motion of the body itself, the former that of the nodes of its orbit. The latter is strictly true only for very great values of β; the former, rightly interpreted, is always true, though it has no special applicability except for (as in the gyroscope pendulum) small values of β. The harmony of the two expressions is easily shown.

Practically the gyroscope is made up of not only a rotating disk, but a non-rotating frame. In estimating the "deflecting force," therefore, in the expression (α) C should apply to the disk alone, and M and γ to the entire mass of disk frame and stem.

The solutions that have been given of equations $(4)_2$ for the freely suspended pendulum are restricted to very small motions; the following is general.

Transfer equations (4) to polar co-ordinates by substituting for x, y, z the values[1]

$$x = l \sin \phi \sin \theta$$
$$y = l \cos \phi \sin \theta$$
$$z = l \cos \theta$$

in which ϕ denotes the azimuth of the pendulum measured from the north, and θ its deviation from the vertical, and we get

$$\frac{d\phi}{dt} = n \sin \lambda + \frac{C}{l^2 \sin^2 \theta} - \frac{2n \cos \lambda}{\sin^2 \theta} \int \cos \phi \sin^2 \theta \, d\theta \tag{7}$$

in which C is a constant depending on arbitrary initial values of $\frac{d\phi}{dt}$, the final term corresponding to the last term of (4).

At the equator we have $\lambda = 0$, and the azimuthal velocity expressed by the third term of (7) becomes $-\frac{2n}{\sin^2 \theta} \int \cos \phi \sin^2 \theta \, d\theta$, which being periodic, produces but very minute change in the plane of vibration. If the pendulum is propelled, from a state of rest in the vertical, in the direction measured by the angle ϕ from the meridian, this angle will be but very slightly affected by the minute values of the above expression during the *outward* excursion, and the increment which $\frac{d\phi}{dt}$ receives will be *almost* exactly neutralized (*quite* so if $\cos \phi$ were absolutely invariable) during the return, and the angular velocity due to the term will again become zero; which cannot happen unless the pendulum *again* pass through the vertical on its return, in which case ϕ will be as little varied during the return; (*otherwise* ϕ will, during the return, pass through all possible values from 0 to $\frac{1}{2}\pi$, and integration is impracticable). Hence we may assume ϕ as constant, and, as in any other latitude, the term in question is, multiplied by $\cos \lambda$, the same as at the equator, we may generally integrate that term *for plane vibrations*, considering ϕ constant, and putting $C = O$. Equation 7 thus becomes,

$$\frac{d\phi}{dt} = n \sin \lambda - n \cos \lambda \frac{\theta - \sin \theta \cos \theta}{\sin^2 \theta} \tag{8}$$

---

[1] The following analysis, as far as equation (9), is modified from Galbraith and Houghton. Proc. R. I. Acad.

If the amplitudes of the vibration are very minute, so that θ may be substituted for $\sin\theta$ in (7), the integral becomes

$$\frac{d\phi}{dt}=n\sin\lambda-\frac{2}{3}n\cos\lambda\cos\phi\,\theta \tag{9}$$

expressing the precession in azimuth, $n\sin\lambda$, precisely as it results from the former analysis (equation 6). The slight periodic disturbance expressed by the second term of (9) escaped that analysis, however, owing to the omission of the terms involving dz.

In the above integrals the angle ϕ must be taken at 180° greater or less on one side of the vertical than on the other, and the parts of $\frac{d\phi}{dt}$ expressed by it will have contrary signs. Hence the curve described in each complete excursion, disregarding the superadded uniform azimuthal motion expressed by the first term of the second member of (9), will have the form of an excessively attenuated leminiscate, or figure of 8.[1]

For greater amplitudes equations (8) will apply until θ becomes nearly equal to 180°; if θ equals or exceeds 180°, it *cannot be assumed* that the pendulum will pass through the zenith (the condition for ϕ to remain nearly constant), and the integral becomes inapplicable and erroneous.

The foregoing integrals involve the condition that the pendulum shall pass through the vertical, and imply that vibration is induced by propulsion from a state of rest in the vertical. But, in the usual form of the experiment for exhibiting the rotation of the earth, the pendulum starts from a state of *relative* rest at the extremity of the initial vibratory arc.

If we disregard the symbolic integral of (7), as may be done, since the minute periodic disturbance it measures has no influence upon the permanent azimuthal motion, that equation will become

$$\frac{d\phi}{dt}=n\sin\lambda+\frac{C}{l^2\sin^2\theta} \tag{a}$$

The second term of the second member is identically the equation of the "spherical pendulum." The latter, we know, exhibits an azimuthal motion of the apsides of its orbit, very minute when C is small, but incomparably greater than the horary azimuthal motion when, C being large, the conjugate dimensions of the orbit approaches equality to the transverse, and of which the limit corresponding to perfect equality of these dimensions is for one vibration,

$$\pi\left[\frac{2}{\sqrt{1+3\cos^2\theta_2}}-1\right] \tag{a)$_2$}$$

as may be deduced from the expression for U_π, par. 731, Peirce, Analyt. Mech., or from expression [79] and [83] of Méc. Cél. (Bowditch), by making $a=b$ and determining the corresponding values of c and dt.

In the case under consideration the value of C will be determined by making $\frac{d\phi}{dt}=0$ for the commencement of motion, $\theta=\vartheta_2$, hence

$$C=-n\,l^2\sin\lambda\sin^2\theta_2$$

[1] See Additional Notes, p. 51.

The pendulum will not move in a plane passing through the vertical, but on a conical surface differing slightly from such a plane, and there will ensue a slight apsidal motion reverse to, and diminishing, the apparent horary motion.

(*b*) The equation $\frac{d\phi}{dt}=\frac{C}{l^2\sin^2\theta}=\frac{C}{l^2-z^2}$ is usually solved by the aid of elliptic integrals (*vide* Prof. Peirce's Analyt. Mech., p. 418); but for present objects the ordinary processes of integration are preferable.

From equations $(3)_2$ and (4) may easily be deduced, neglecting terms containing n, and bearing in mind that

$$x^2+y^2=l^2-z^2,\ x\,dx+y\,dy=-z\,dz,$$

$$(c)\qquad dt=\frac{-l\,dz}{\mp\sqrt{(l^2-z^2)(c+2gz)-C^2}}$$

in which the upper or lower sign of the radical is to be taken according as dz is positive or negative, that is, as the pendulum is descending or ascending. The quantity under the radical may be put in the form (*vide* Mec. Celeste, Bowditch, Vol. I. p. 52).

$$2g\,(a-z)\,(z-b)\left(z+\frac{f}{2g}\right)$$

in which

$$\frac{f}{2g}=\frac{l^2+a\,b}{a+b}$$

$$(d)\qquad C^2=\frac{2g\,(l^2-a^2)\,(l^2-b^2)}{a+b}$$

$$c=2g\frac{(l^2-a^2-ab-b^2)}{a+b}$$

a and b being the greatest and least values of z.

If now we transfer the origin of co-ordinates to the lowest point of the spherical surface by substituting for z, a, and b, $l-u$, $l-\alpha$, $l-\beta$, and replace C and dt in (*b*) by the values above found, we shall have (putting $l+\frac{f}{2g}=\rho$)

$$d\phi=\tfrac{1}{2}\frac{\sqrt{(l^2-a^2)\,(l^2-b^2)}}{a+b}\,\frac{l\,du}{u(2l-u)\,[(\rho-u)\,(u-\alpha)\,(\beta-u)]^{\frac{1}{2}}}$$

the varying sign of the radical being understood. If we develop the two factors $(2l-u)^{-1}$ and $(\rho-u)^{-\frac{1}{2}}$, and multiply the results, we shall have

$$d\phi=\sqrt{\frac{(l^2-a^2)(l^2-b^2)}{a+b}}\int\frac{l\,du}{u[-\alpha\beta+(\alpha+\beta)\,u-u^2]^{\frac{1}{2}}}\left[\frac{1}{2l\rho^{\frac{1}{2}}}-\frac{1}{4l\rho^{\frac{1}{2}}}\left(\frac{1}{l}+\frac{1}{\rho}\right)u\right.$$

$$\left.+\frac{1}{8}\frac{1}{l\rho^{\frac{1}{2}}}\left(\frac{1}{l^2}+\frac{1}{l\rho}+\frac{3}{2\rho^2}\right)u^2+\frac{1}{16l\rho^{\frac{1}{2}}}\left(\frac{1}{l^3}+\frac{1}{l^2\rho}+\frac{3}{2l\rho^2}+\frac{15}{6\rho^3}\right)u^3+\&c.\right]$$

Strike out the common factor l, and remove the factor $2\rho^{\frac{1}{2}}$ into the denominator of the first radical factor (which factor then becomes $\sqrt{(l-a)(l-b)}=\sqrt{\alpha\beta}$), and the above integral becomes (*vide* Hirsch, Integral Tables, pp. 160–164), writing U for $-\alpha\beta+(\alpha+\beta)\,u-u^2$,

4 January, 1872.

$$\phi=\tfrac{1}{2}\cos^{-1}\Big(\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2-4\alpha\beta}}\cdot\frac{\sqrt{U}}{u}\Big)+\frac{1}{4}\sqrt{\alpha\beta}\Big(\frac{1}{l}+\frac{1}{\rho}\Big)\cos^{-1}\frac{2\sqrt{U}}{\sqrt{(\alpha+\beta)^2-4\alpha\beta}}$$
$$+\frac{1}{8}\sqrt{\alpha\beta}\Big(\frac{1}{l^2}+\frac{1}{l\rho}+\frac{3}{2\rho^2}\Big)\Big[-\sqrt{U}+\frac{1}{2}(\alpha+\beta)\cos^{-1}\frac{2\sqrt{U}}{\sqrt{(\alpha+\beta)^2-4\alpha\beta}}\Big]$$
$$(e)\qquad+\frac{1}{16}\sqrt{\alpha\beta}\Big(\frac{1}{l^3}+\frac{1}{l^2\rho}+\frac{3}{2l\rho^2}+\frac{15}{16\rho^3}\Big)\Big[-\Big(-\frac{1}{2}u+\frac{3}{4}(\alpha+\beta)\Big)\sqrt{U}+$$
$$\Big(\frac{3}{8}(\alpha+\beta)^2-\frac{1}{2}\alpha\beta\Big)\cos^{-1}\frac{2\sqrt{U}}{\sqrt{(\alpha+\beta)^2-4\alpha\beta}}\Big]+\&c.,+\text{cons't.}$$

In order that the arcs in the above expression should continually increase with the time, the positive or negative sign must be applied to the radical $\sqrt{U}$ according as u is increasing or diminishing: taken from $u=\alpha$ to $u=\beta$ (or the converse), the arcs all become $=\pi$, and the non-circular functions vanish.

Hence the azimuthal angle passed over by the pendulum in its motion from a lowest to a highest point of its orbit (or the converse) is expressed by

$$\phi_{\frac{1}{2}\pi}=\frac{1}{2}\pi\Big[1+\frac{1}{2}\sqrt{\alpha\beta}\Big(\frac{1}{l}+\frac{1}{\rho}\Big)+\frac{1}{4}\sqrt{\alpha\beta}\Big(\frac{1}{l^2}+\frac{1}{l\rho}+\frac{3}{2\rho^2}\Big)\frac{\alpha+\beta}{2}$$
$$(f)\qquad+\frac{1}{8}\sqrt{\alpha\beta}\Big(\frac{1}{l^3}+\frac{1}{l^2\rho}+\frac{3}{2l\rho^2}+\frac{15}{6\rho^3}\Big)\Big(\frac{3(\alpha+\beta)^2}{8}-\frac{1}{2}\alpha\beta\Big)$$
$$+\frac{1}{16}\sqrt{\alpha\beta}\Big(\frac{1}{l^4}+\frac{1}{l^3\rho}+\frac{3}{2l^2\rho^2}+\frac{15}{6l\rho^3}+\frac{105}{24\rho^4}\Big)\Big(\frac{5(\alpha+\beta)^3}{16}-\frac{3\alpha\beta(\alpha+\beta)}{4}\Big)+\&c.\Big]$$

The sum of the terms after unity included in the brackets is the ratio by which the azimuth angle exceeds a quadrant; or, if the integral is taken through an entire revolution (relatively *to the apsides*), it, multiplied by 2π, is angle of advance of the apsides per revolution.

For motion nearly oscillatory, of whatever amplitude (*i. e.*, α being small and β arbitrarily large), or for spherical motions of considerable amplitude (α and β taken within limits not exceeding say one-third of l, corresponding to a swing of over 90°), ρ, always greater than $2l$, differs but slightly from that magnitude. Giving ρ that value, and taking the angle ϕ for a complete vibration, or a semi-apsidal revolution, we have the formula,

$$(g)\qquad\phi_\pi=\pi\Big[1+\frac{3}{2}\cdot\frac{\sqrt{\alpha\beta}}{2l}+\frac{1.3.5}{1.2.4}\cdot\frac{\sqrt{\alpha\beta}.\frac{1}{2}(\alpha+\beta)}{(2l)^2}+\&c.\Big]$$

If α and β are both small and nearly equal, and θ the angle of which they are the versed sine, then $\frac{\sqrt{\alpha\beta}}{l}=\frac{1}{2}\sin^2\theta$, and the apsidal motion corresponding to the second term of the above (the following terms neglected) becomes $\frac{3}{8}\pi\sin^2\theta$; agreeing with the expression $(a)_2$ on p. 24, when developed for the same case.

If the pendulum moves *nearly* horizontally in a great circle, that is, if $\alpha+\beta=2l$ and $\alpha\beta=l^2$ (nearly), then ρ, C, and c are each infinitely great, and (f) becomes

$$\phi_{\frac{1}{2}\pi}=\tfrac{1}{2}\pi\Big[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{32}+\&c.\Big]=\pi$$

which denotes that the line of apsides moves through a quadrant while the pendulum is passing, through 180° azimuth, from a highest to a lowest point; in other words, that the highest and lowest points are diametrically opposite, and the apsides are *apparently* stationary. This theorem is true (as shown by Prof. Peirce), whatever be the inclination of the great circle, though it cannot be *generally* made evident by the above formulæ.

If in (*c*) we make transformations and substitutions already described, develop the factor $(\rho-u)^{-\frac{1}{2}}$, integrate between the limits $u=\alpha$, $u=\beta$, and double the result, we shall have for the time of one vibration, or one semi-orbital revolution,

$$(h)\quad T=2\pi\frac{l}{\sqrt{2g\rho}}\left[1+(\tfrac{1}{2})^2\frac{\alpha+\beta}{\rho}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{\alpha+\beta}{\rho^2}\right)^2-\frac{1.3}{2.4}\,\frac{\alpha\beta}{2\rho^2}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{(\alpha+\beta)^3}{\rho^3}\right.$$
$$\left.-\frac{1.3.5}{2.4.6}\cdot\frac{3\alpha\beta(\alpha+\beta)}{4\rho^3}+\&c.\right]$$

and if $\alpha=0$, that is, if the motion is purely oscillatory, then $\rho=2l$, and this becomes

$$(i)\qquad T=\pi\sqrt{\frac{l}{g}}\left[1+(\tfrac{1}{2})^2\frac{\beta}{2l}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{\beta}{2l}\right)^2+\&c.\right]$$

and more generally for any value of α and β not exceeding the versed sine of thirty or forty degrees:

$$(j)\qquad T=\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2\frac{\alpha+\beta}{2l}-\frac{1.3}{2.4}\,\frac{\alpha\beta}{(2l)^2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{\alpha+\beta}{2l}\right)^2\pm\&c.\right]$$

When α and β are both small, as in most pendulum experiments, the terms after the first in the brackets may be omitted, and we have the ordinary expression for the time $\pi\sqrt{\frac{l}{g}}$.

In the expression (*g*), omitting all the terms in brackets after the second, it is evident that that term will measure the apsidal motion for the time $\pi\sqrt{\frac{l}{g}}$; and hence that the total integral of equation (*a*) taken through that time will be,

$$\pi\left[n\sin\lambda\sqrt{\frac{l}{g}}\pm\frac{3\sqrt{\alpha\beta}}{4l}\right]$$

and if we denote by ϕ' the angle of azimuth of the apsidal line measured from its initial direction, we shall have, at any time t, substituting for α and β, $l\,(1-\cos\theta_1)$ and $l\,(1-\cos\theta_2)$,

$$(k)\qquad \phi'=\left[n\sin\lambda\pm\frac{3}{4}\sqrt{\frac{g}{l}(1-\cos\theta_1)\,(1-\cos\theta_2)}\right]t$$

or, since θ_1 is always small,

$$k')\qquad \phi'=\left[n\sin\lambda\pm\frac{3}{4}\sqrt{\frac{g\,(1-\cos\theta_2}{2l}}\sin\theta_1\right]t$$

The angle θ_2 being given, θ_1 is determined for the *ordinary* pendulum experiment by the consideration that the constant C, of which the value is found p. 24, is the moment of the quantity of motion. As c is very minute, the actual velocity at the

lowest point will be, sensibly, $\sqrt{2gl(1-\cos\theta_2)}$, and hence the moment at that point will be $l \sin\theta_1 \sqrt{2gl(1-\cos\theta_2)} = C = -nl^2 \sin\lambda \sin^2\theta_2$, from which deducing the value of $\sin\theta_1$ and substituting in (k'), we have

$$(l) \qquad \phi' = n \sin\lambda \left(1 - \frac{3}{8}\sin^2\theta_2\right) t$$

The second term in brackets is the retardation from the true horary motion; it being implied, of course, that the amplitude of oscillation is preserved unimpaired. Though small, this retardation would be sensible, especially if θ_2 had a considerable magnitude, say eight or ten degrees, though inappreciable for very small oscillations.

Practically, the resistance of the air constantly diminishes the value of θ_2, and failure to procure a perfect state of rest to the pendulum before it is set free, or currents of air, may give quite different values to C and c, and determine the character of the orbital motion to be progressive instead of retrograde; and it is generally observed that the conjugate dimension of the orbit increases (probably owing to the resistance of the air) as θ_2 diminishes. In this way the apsidal motion due to the orbit may acquire a value quite considerable compared to the proper horary motion, which will apparently be sensibly retarded or accelerated. By observing, at any period of the experiment, the value of θ_1 and θ_2 and the direction of the orbital motion, the coefficient of t in the formula (k') will give the theoretical rate of azimuthal motion at that instant.

If the orbital motion is retrograde and

$$(m) \qquad \sin\theta_1 = \frac{4}{3} n \sin\lambda \sqrt{\frac{2l}{g(1-\cos\theta_2)}}$$

the line of the apsides would be stationary. For Columbia College, where $n \sin\lambda = .00004747$, with a pendulum of 26 feet in length, and a value of θ_2 of 6°, this would give $\theta_1 = 3',3$, the actual semi-axes of the projection of the orbit being about two feet nine inches and one-third of an inch.

It would generally be sufficiently accurate to substitute for $1-\cos\theta_2$, $\frac{1}{2}\sin^2\theta_2$, by which formula ($k'$) would become more simply,

$$(n) \qquad \phi' = \left[n \sin\lambda \pm \frac{3}{8}\sqrt{\frac{g}{l}} \sin\theta_1 \sin\theta_2 \right] t$$

in which it is seen that the deviation from the proper horary motion is proportional to the area of the projection of the orbit.

The above, or (k'), expresses the azimuthal motion as it would be were there no other forces acting than those included in the investigation, in which case θ_1 and θ_2 would be invariable. In point of fact there are practically numerous disturbing forces, of which, however, the resistance of the air is the most considerable, and through which θ_1 and θ_2 are incessantly changing, and it is not improbable that the *change of shape* of the orbit may, in itself, cause some variation of direction of the line of apsides:[1] a matter which cannot be decided until the problem of the spherical pendulum is solved with the resistance taken into account.

---

[1] An investigation of the simpler case of the plane elliptical motion of a body attracted by a central force proportional to the distance, in a medium which resists either directly or as the square

To get a clearer idea of what is expressed by the periodic term of (7), $-2n \cos\lambda \int \cos\phi \sin^2\theta\, d\theta$ (which corresponds to the integral $\int y\, dz$ of (4)), we must revert to the latter equation. Conceive the pendulum propelled from a state of rest in the vertical, with a very great angular velocity, denoted by v, in the plane of the meridian. Were the earth motionless, it would continue to whirl in this plane, passing through the zenith at every revolution. Introduce the element of the earth's rotation, and the two terms of equation (4) containing n take effect, by the first of which the plane of revolution moves in azimuth with the angular velocity $n \sin\lambda$. The second expressing that there will be an increase of the moment of the quantity of motion about the vertical after a time T proportionate to the area generated in that time $\int y\, dz$. Under these conditions this area is cumulative, and at the end of one revolution expresses the area of the circle of radius l. Let us suppose that the plane of motion turns about a line parallel to the complementary terrestrial axis with an angular velocity $n \cos\lambda$. At the end of the time T (supposed very small) the plane will make with the meridian the angle $n \cos\lambda T$, and as the quantity of motion in its own plane is vl, its moment referred to a vertical axis will, from zero, have become $vl^2 \sin(n \cos\lambda T)$, or, substituting the small arc for its sine,

$$vl^2\, n \cos\lambda T$$

But T, for one revolution, is expressed by $\frac{2\pi}{v}$ hence the above becomes

$$2n \cos\lambda\, \pi\, l^2$$

The area of the circle which is generated in the same time is πl^2 and is expressed by the integral $\int y\, dz$, and it is easy to show for each successive revolution that the area $\int y\, dz$ multiplied by $2n \cos\lambda$ corresponds to an increment of the moment of quantity of motion about the vertical which it would receive from a turning of the plane about the complementary axis through the angle $n \cos\lambda\, t$.

Hence, for the particular case under consideration, the second term of second member of equation 4 expresses an angular motion about the complementary axis of which $n \cos\lambda$ is the velocity. The resultant of this, and the azimuthal component, is rotation about an axis parallel to that of the earth, and opposite in direction to the earth's rotation.

The above theorem can be analytically demonstrated. The quantity N, expressive of the tension of the cord, is made up of the centrifugal force due to the pendulum's relative angular motion and of the variable component of the force of gravity (neglecting, as we have done, quantities of the order n^2). If this centrifugal force is so great that the component of gravity may be neglected, $\frac{N}{l}$ will

---

of the velocity, indicates no apsidal motion accompanying the decrease of parameters of the orbit. Neither, however, does it indicate the enlargement of the minor axis (initially very small) so universally observed in the pendulum experiments.

reduce to the constant v^2 (v being the impressed angular velocity of pendulum movement), and the second and third of equations (3) will yield the equation

$$\frac{d^2}{dt^2}(y \cos \lambda - z \sin \lambda) = -v^2 (y \cos \lambda - z \sin \lambda) - g \sin \lambda$$

the integral of which is

$$y \cos \lambda - z \sin \lambda = -\frac{g \sin \lambda}{v^2} + A \cos vt + B \sin vt$$

The above is independent of n; $y \cos \lambda - z \sin \lambda$ is the value of the new co-ordinate of y' when the axis of y is changed to parallelism to that of the earth. Hence, as thus transformed, this co-ordinate is unaffected by the earth's rotation, the plane of pendulum motion must turn (if it turns at all) about such an axis.

The assumption for $t=0$, of $y=0$, $z=l$, $\frac{dy}{dt}=lv$, $\frac{dz}{dt}=0$, gives

$$\begin{cases} A = -\left(l - \frac{g}{v^2}\right) \sin \lambda \\ B = l \cos \lambda \end{cases}$$

(10) hence $$y \cos \lambda - z \sin \lambda = -\frac{g}{v^2} \sin \lambda \, (1 - \cos vt) + l \sin (vt - \lambda)$$

The second term of the second member of the above gives precisely the value which the first member would have, were the plane of pendulum motion stationary or were it turning with *any* angular velocity about an axis parallel to the earth's. The first term is, owing to the assumed high value of v, very minute, and is *periodic*, the period being equal to $\frac{2\pi}{v}$, the time of the pendulum's revolution in its circular orbit. Owing to its minuteness and periodicity, this term may be neglected, and equation (10) becomes

$$y \cos \lambda - z \sin \lambda = l \sin (vt - \lambda)$$

to satisfy which, and at the same time the three differential equations (3) (omitting g, as we have found reason to do in (10), and also omitting terms containing n_2, in the developments), requires the following values for the co-ordinates:—

$$\begin{aligned} x &= l \sin nt \cos (vt - \lambda) \\ y &= l \left[\cos \lambda \sin (vt - \lambda) + \sin \lambda \cos nt \cos (vt - \lambda)\right] \\ z &= l \left[-\sin \lambda \sin (vt - \lambda) + \cos \lambda \cos nt \cos (vt - \lambda)\right] \end{aligned}$$

Changing the axes of y and z by turning them through the angle λ, we should have for new co-ordinates,

$$\begin{aligned} x' &= l \sin nt \cos (vt - \lambda) \\ y' &= l \sin (vt - \lambda) \\ z' &= l \cos nt \cos (vt - \lambda) \end{aligned}$$

If we now change the plane of yz by moving it through the variable angle nt about the axis of y', we get,

$$\begin{aligned} x'' &= 0 \\ y'' &= l \sin (vt - \lambda) \\ z'' &= l \cos (vt - \lambda) \\ y''^2 + z''^2 &= l^2 \end{aligned}$$

These values show that the plane of pendulum revolution turns about an axis parallel to the earth's with the relative angular velocity n; or, in other words, that the plane preserves its parallelism to itself in space.

If we had a *succession of pendulums* rigidly connected with each other, the disturbing effect of gravity would be eliminated. Such a succession would be simply a *gyroscope*, and the gyroscope, mounted in gimbals and set running in a meridian plane, would exhibit the apparent rotation of its disk around an axis parallel to the earth's axis equal and contrary to that of the earth; which is simply saying that as the earth revolves the plane of the disk maintains its parallelism to itself, and if we suppose its axis directed at a star in the plane of the equator, it would follow that star so long as the rotation of the disk is sustained.[1]

The pendulum experiment, in its ordinary form, exhibits not the whole rotation of the earth, but only one component of it; the component which belongs to an axis passing through the locality. It is perhaps quite as interesting and important, as being the only experimental demonstration we can have of a principle difficult of comprehension, but as fundamental to mechanics, since its enunciation by Euler, as the corresponding one of the decomposition of linear velocities, viz., that of the decomposition into distinct components, of rotary velocities. The plane of the pendulum appears to turn relatively to the surface of the earth simply because the earth turns just so much underneath it, the earth really *revolving* about the local axis with a certain calculable component of velocity. The earth turns at the same time with another component of velocity about another axis (the complementary one), and the joint effect, or the resultant of the two components, is the rotation about the polar axis. The second component, very great as we approach the equator, where the first vanishes entirely, is not exhibited by the pendulum, and is only detected by analysis as a slight disturbance. Convert the pendulum into the gyroscope, however, and this second component appears equally with the first.

---

[1] Owing to the friction of the gimbals, there would be, *practically*, besides the motion above described, a motion of the axis in the plane of the meridian, north or south, according to the direction of the disk rotation; this angular motion might be greater or less than the equatorial motion, but would be, with a well-constructed apparatus, independent of it, at least for the brief time during which a gyroscope experiment would last.

ON THE

INTERNAL STRUCTURE OF THE EARTH CONSIDERED AS AFFECTING THE PHENOMENA OF PRECESSION AND NUTATION.

The equations of precession and nutation are, as is well known, entirely independent of any particular law of density, and are functions only of the absolute values of the moments of inertia about the equatorial and polar axes A and C,[1] and are independent indeed of the figure of the earth, except so far as it affects the values of these moments.

Moreover, if the earth, instead of being solid throughout, is (as supposed by most geologists) a solid shell inclosing a fluid nucleus, it is only necessary (leaving out of consideration the pressure that may be exerted on the interior surface by the fluid) that the shell should have these moments of inertia. Mr. Poinsot[2] has obtained as the results of calculation for a homogeneous spheroid, values of precession and nutation identical with those of observation, by taking the ellipticity at $\frac{1}{308.65}$ and η (the ratio of mass of moon to that of the earth), at $\frac{1}{88}$.

We have, assuming a uniform density, indicating by a and b the equatorial and axial radii, and by e the ellipticity:—

$$C=\frac{8}{15}\pi a^4 b=(\text{approx.})\ \frac{8}{15}\pi b^5(1+4e)$$

$$A=\frac{4}{15}\pi(a^4 b+a^2 b^3)=\frac{8}{15}\pi b^5(1+3e)$$

$$\frac{C-A}{C}=\frac{e}{1+4e}=e-4e^2 \tag{46}$$

If e is taken at $\frac{1}{308.65}$ then $\frac{C-A}{C}=\frac{1}{312.7}$. But all meridian measurements of

---

[1] I assume, of course, the equality of all moments of inertia, A, about the equatorial axes, and overlook all questions as to the non-symmetry of the earth with respect to its axis of figure or to the equator; for, in fact, neither the rotation of the earth nor any observable celestial phenomena reveal it.

[2] Connaissance des temps, 1858.

the earth indicate an ellipticity greater rather than less than $\frac{1}{300}$;* and the latest determination makes it $\frac{1}{295}$,† by giving which value to e we obtain $\frac{C-A}{C}=\frac{1}{299}$.

Therefore the observed precession is to that which would result in a homogeneous spheroid, from the formulas, with the latest determined value of e introduced, as 299 : 311.7, provided the relative mass of the moon be but $\frac{1}{88}$.‡

The value of $\frac{C-A}{C}$ would be the same for a homogeneous *shell* of which the interior surface had the *same ellipticity*, e, as the exterior; or it would be the same for a shell of which all the elementary strata had the same ellipticity, in which the density, constant through each stratum, should vary according to any law, from stratum to stratum. The ratio of 299 : 312.7, so nearly unity ($=0.96=\frac{24}{25}$ nearly), while the ratio of mean to surface density of the earth is so high, indicates nearly uniform ellipticity of stratification, and hence fluidity of *origin;* while, on the other hand, the considerable inequalities in the equatorial axes indicated in the note below are incompatible with the hypothesis of *actual* fluidity beneath a *thin* crust, and are, to the measure of their probability, a disproof of it.

The effect upon the axial movements of such a shell which would result from the pressures of an internal fluid has been made the subject of an elegant mathematical investigation by W. Hopkins, F.R.S., in the Philosophical Transactions of 1839–40–42. On the supposition of a uniform density of shell and fluid, and the same ellipticity for inner and outer surfaces of the shell, the precession will be the same

---

* Airy, "Figure of the Earth," Encyc. Metrop.; Guillemin: Mädler, Am. Journ. of Science, Vol. 30, 1860, makes the polar compression of greatest meridian $\frac{1}{292.109}$

of smallest meridian $\frac{1}{302.004}$

(Article translated by C. A. Schott, U.S. Coast Survey, from Prof. Heis' "Astronomie, Météorologie et Géographie," Nos. 51, 52. 1859.)

† Appendix "Figure of the Earth" to the "Comparisons of Standards of Length," published 1866 by the British Ordnance Survey, gives for a "spheroid of revolution," $\frac{a-c}{c}=\frac{1}{295}$; for a spheroid of three axes, $\frac{a-c}{c}=\frac{1}{285.97}$, $\frac{b-c}{c}=\frac{1}{313.38}$, $\frac{a-b}{c}=\frac{1}{3269.5}$. The probabilities of the latter supposition to the former being 154 : 138.

‡ There is yet great uncertainty as to the relative mass of the moon, and as long as that point is unsettled, so is also the ratio of observed to calculated precession. Laplace, from observations of the tides at Brest, fixed it at $\frac{1}{75}$, which number is adopted by Pontécoulant. Former determinations from the observed nutations make it $\frac{1}{80.75}$; but $\frac{1}{87}$ was the determination from the coefficient of nutation of Lindenau. Guillemin gives $\frac{1}{88}$, and these two last numbers coincide nearly with that used by Poinsot. A discussion by Mr. Wm. Ferrell, member of National Academy of Sciences, of tidal observations made for a series of years at the port of Boston, as well as those at Brest, gives results confirmatory of the larger ratio of Laplace. Serret (Annales de l'Observatoire Imp. 1859) assumes $\frac{1}{83}$ and deduces $\frac{C-A}{C}=\frac{1}{306}=0.00327$. These ratios are adopted by Thomson and Tait, §§ 803, 828. Archdeacon Pratt ("Figure of the Earth," 4th ed. 1871) adheres to Laplace's determination.

and the nutation essentially the same as for a homogeneous spheroid; but for the actual case of a heterogeneous fluid contained in a heterogeneous shell he finds that the ellipticity of the inner surface of the shell must be less than that of the exterior in the proportion of the *observed* precession to the precession of a homogeneous spheroid of same external ellipticity, a proportion which he assumes to be $\frac{7}{8}$, in accordance to the then received numbers for ellipticity, and for the mass of the moon.

The fulfilment of the condition, in the actual constitution of the earth, is improbable; for the isothermal surfaces are in all probability (and indeed by his own mathematical conclusions) of progressively *greater* ellipticity from the surface inwards. He finds, however, the requisite decrease of ellipticity in the fluid surfaces of *equal density*, assuming the well-known hypothetical law of density of Laplace $\rho = A \frac{\sin qb}{b}$; but we know not the influence which pressure has upon solidification, and it seems probable that the interior surface of the shell would conform nearly to the surfaces of equal temperature. The demand which he makes for 800 or 1000 miles thickness of shell is therefore a minimum (for the data used), while the more *probable* result is a much greater thickness or even entire solidity. But however elegant may be Mr. Hopkins' analysis, the basis of the structure is but slender, and those results have not been generally accepted as fully decisive of the question.

Sir Wm. Thomson, F.R.S., brings forward (Philosophical Transactions, 1863; also Thomson's and Tait's Treatise on Natural Philosophy, 1867) arguments against the popular theory of a thin crust, which are more forcible. A thin crust would itself undergo tidal distortion, and the height of the *apparent* tides of the ocean be thereby much reduced, while the *actual* precession would be diminished in the same ratio; that is, the differential forces of the sun and moon would expend themselves in producing these solid tides instead of producing precession.

From a theoretical investigation (given in a separate paper in the same volume)[1] of the deformation experienced by a homogeneous elastic spheroid under the influence of any external attracting force, he arrives at the result that, if the earth had no greater rigidity than steel or iron, it would yield about $\frac{2}{5}$ as much to tide-producing influences as if it had no rigidity—more than $\frac{3}{4}$ as much if its rigidity did not exceed that of glass. Moreover, the apparent ocean tides (or difference of high and low water level) would be (if *H* is the measure for a perfectly rigid earth) 0.59 *H*, if the earth had the rigidity of iron or steel only; $\frac{1}{4\frac{1}{2}}$H if it had that of glass.

As to *precession*, the centrifugal force of the crowns of the tidal elongation would balance $\frac{1}{9}$ of the dynamic couple resulting from the sun's or moon's attraction if the earth had only the rigidity of glass, and $\frac{2}{5}$ if it had only that of steel.

"That the effective tidal rigidity, and what we call the precessional effective

[1] See also "Treatise on Natural Philos.," § 832 *et seq.*

rigidity of the earth, may be several times as much as that of iron (which would make the phenomena, both of the tides and precession, sensibly the same as if the earth were perfectly rigid), it is enough that the actual rigidity should be several times as great as the actual rigidity of iron throughout 2000 or more miles thickness of crust."

A theorem fundamental to the establishment of the above propositions is, that a revolving spheroid destitute of rigidity, a homogeneous fluid one, for instance, would have no precession. Sir W. Thomson does not mathematically demonstrate this theorem, but by use of an hypothesis gives an elegant illustration of its truth, for which, though to me it is convincing, I prefer to substitute the following demonstration.

Such a spheroid, all the particles of which revolve about an axis with a common angular velocity n, and attract each other by the law of universal gravitation, would have the form of an ellipsoid of revolution, the ellipticity of its meridional section being $\frac{5}{4}\frac{n^2}{g}$.* (See "Figure of the Earth," Encyc. Metrop., par. 33, by Prof. Airy.) Attracted by the sun, its tides would be expressed by the terms of [2316] Méc. Cél., Book IV (Bowditch). Of these three terms, the first (a function of the declination *only* of the attracting body) and the third (the semi-diurnal oscillation) express tidal elevations symmetrically distributed on each side of the equator, which would, hence, exert no influence through the centrifugal forces of their masses, upon precession. The second therefore, or *the diurnal tide*, is alone to be considered.

Conceive a meridian plane passed through the sun at any declination, the "couple" exerted by its attraction would be exerted wholly to turn the spheroid about an equatorial axis normal to this plane. We have therefore to investigate what dynamic couple, with reference to this same axis, will be exerted by the centrifugal force of the diurnal tidal protuberance. As the calculation involves the state of things at but a single instant of time, the angle, $nt+\varpi-\psi$, may be written ϖ and counted from the meridian of the sun: ρ, the uniform density of the fluid, taken as unity. The height, y, of the diurnal tide will be expressed for all parts of the spheroid by

$$y=\frac{15S}{2r^3g}\sin\theta\cos\theta\sin\lambda\cos\lambda\cos\varpi\dagger \tag{47}$$

in which λ is the polar distance or complement of the latitude of the locality, and θ the declination of the sun. If, with Laplace, we put $\cos\lambda=\mu$, and $\sin\lambda=\sqrt{1-\mu^2}$, the mass of the elementary column of height y will be $yd\mu\,d\varpi$, and its centrifugal

---

* g being the force of gravity *at the equator of the hypothetical spheroid.*

† The expression, in the original, for the diurnal oscillation, is

$$\frac{3L}{r^3g\left(1-\frac{3}{5\rho}\right)}\sin V\cos V\sin\theta\cos\theta\cos(nt+\varpi-\psi)$$

The notation of my paper on the precession of the equinoxes is substituted, and the assumed value of ρ introduced.

force (the radius of the spheroid being taken at unity, and the variation, assumed slight, due to ellipticity, disregarded) $n^2 y\, d\mu\, d\varpi \sqrt{1-\mu^2}$. The component of this tending to tilt the spheroid about the axis in question is $n^2 y\, d\mu\, d\varpi \sqrt{1-\mu^2} \cos \varpi$, and its moment $n^2 y\, d\mu\, d\varpi\, \mu \sqrt{1-\mu^2} \cos \varpi$.

Substituting the value of y (47), the above becomes

$$n^2 \frac{15S}{2r^3 g} \sin \theta \cos \theta\, d\mu\, d\varpi\, \mu^2 (1-\mu^2) \cos^2 \varpi$$

Integrating, first with reference to μ from $\mu=-1$ to $\mu=+1$, then with reference to ϖ from 0 to 2π, we get, as the expression for the couple due to "the centrifugal force of the crowns of the tidal elongation," resisting the sun's action,

$$2\pi \frac{n^2 S}{g r^3} \sin \theta \cos \theta \tag{48}$$

We have found (19) for the moment of the sun's force, producing precession, the expression

$$\frac{3S}{r^3} (C-A) \sin \theta \cos \theta$$

and (46), $(C-A)=\frac{8}{15}\pi e$ (b being taken at unity) and e, as already stated, is for a homogeneous fluid spheroid $=\frac{5}{4}\frac{n^2}{g}$. Making these substitutions, the above expression becomes identical with (48). The precessional force of the sun is, therefore, exactly neutralized by the centrifugal force of the tidal swelling.

The theorem could, doubtless, be demonstrated for a revolving fluid spheroid in equilibrium, of which the density of the strata varies. Without extending any further the mathematical analysis, it will be sufficient to remark that the calculation of the tidal elevations is, identically, that of equilibrium of form of the revolving body subjected to a foreign attraction, and in the calculation the motion of rotation is disregarded, and the centrifugal force, which expresses its entire effect upon the form, alone considered. Under this point of view, equilibrium of form is, necessarily, equilibrium (or stability) of *position*. For if any effective turning force exists, it must, in order not to interfere with equilibrium of form, either be so distributed as to give each individual particle of the spheroid its proper relative quantity of turning motion, or it must be a distorting force. The first alternative cannot be admitted; the second is excluded by the hypothesis of equilibrium. Hence, there can be no turning (or precessional) force.

The accuracy of the foregoing analysis is complete,[1] except that the consideration of *relative motion* of the particles is excluded. But Laplace shows (p. 604, Vol. II, Bowditch) that as the depth of the ocean increases, the expressions for the tidal

[1] There are slight errors of approximation: 1st, in the tidal expression (47) itself; 2d, in the above integration which disregards the variation of the radius; and, 3d, in the value of $C-A$. They neutralize each other in the final result.

oscillations given by the dynamic theory approximate rapidly to those of the "equilibrium theory," with which, when the depth is very great, or the spheroid wholly fluid, they are essentially identical. Moreover, he shows (p. 219, Vol. I) that the *vertical* motions of the particles, when the depth is small, may be disregarded. When the spheroid is wholly fluid, *all* the relative motions of the particles are of the same order as the vertical ones and exceedingly minute; and the forces of inertia thereby developed are insensible compared with those we have been considering.[1]

By parity of reasoning the truth of Sir W. Thomson's propositions concerning a solid but yielding spheroid is made evident; for exactly in the same ratio to the tides of a fluid spheroid that the solid tidal elevations are produced (the actual ellipticity of the earth being nearly that of equilibrium with the centrifugal forces), will the precessional couple due to the tide-producing attraction be neutralized by their centrifugal action.[2] That a thin solid crust, such as geologists generally assume, would yield and exhibit tidal elongations, seems without calculation very probable; but if Sir W. Thomson is correct as to the rigidity required in even a wholly solid earth, the hypothesis of a thin crust must be abandoned, and it would seem indeed that rigidity several times as great as the actual rigidity of iron throughout 2000 or more miles thickness of crust would be incompatible with a very high internal temperature.

Without having recourse to Sir W. Thomson's profound analysis, the necessity, in order that there shall be no sensible solid tidal wave, of a very high rigidity

---

[1] The foregoing demonstration does not conflict with Laplace's theorem that ocean tides do not affect the precession; for his theorem applies only to a shallow ocean over a rigid nucleus, of which ocean the precessional couple, by altered attractions, pressures, and centrifugal forces due to generation of living forces in the fluid, is transferred to the nucleus. I have already alluded to the minuteness of the motions of the particles of a fluid spheroid. The remarks apply, *à fortiori*, to those of an elastic *solid*. *Vibratory* motions, properly speaking, cannot exist, for the elastic forces extremely minute are always held (sensibly) in equilibrium by the distorting forces. The solid surface would oscillate in the same sense that the ocean tides oscillate, *i. e.*, by a "forced" tide-wave.

[2] "It is interesting to remark," say Thomson and Tait (§ 848, "Treatise, &c."), "that the popular geological hypothesis of a thin shell of solid material, having a hollow space within it filled with liquid, involves two effects of deviation from perfect rigidity which would influence in opposite ways the amount of precession. The comparatively easy yielding of the shell must render the effective moving couple due to sun and moon much smaller than it would be if the whole interior were solid, and, on this account, must tend to diminish the amount of precession and nutation. But the effective moment of inertia of a thin solid shell, containing fluid in its interior, would be much less than that of the whole mass if solid throughout; and the tendency would be to much greater amounts of precession and nutation on this account."

The co-efficient of precession of the "thin solid shell" would be (p. 34) the same, nearly, as that of the spheroid of which the homogeneous strata have the same ellipticity. Its precession-resisting couple (48) due to tidal distortion would be just what is necessary to develop its proportional influence upon the precession *of that shell*, upon which the fluid contents can exert influence only through their pressure. This is identically Prof. Hopkins' problem. The *thin* shell of popular geological hypothesis would, however, be subject to tidal distortions scarcely inferior in magnitude to those of a wholly fluid spheroid; by which, as we have seen, the sun and moon's "moving-couple" is wholly neutralized throughout the whole spheroid.

for the earth, may be made evident from the following considerations: A rod of steel extending towards the sun from the centre to the surface of the earth, would be elongated by the differential force of the sun's attraction $0^{ft}.975$, or one foot, nearly. The height of the solar tide of a homogeneous fluid spheroid is $1^{ft}.355$; but the *mutual attraction* of the elevated particles produces $0^{ft}.793$ of this, and the remaining $0^{ft}.542$ is the proper measure of the *direct* action of the solar force. In the case of the rod the elastic forces of the steel *alone* are considered; in the spheroid gravitation is the sole binding force. The maximum extension of the rod per unit of length would be expressed by the decimal .000000055 corresponding to a tensile force of 1.87 lbs. (taking the coefficient of elasticity at 34 millions lbs.) per square inch.(1) The necessity of the extreme rigidity demanded by Sir W. Thomson is recognized when it is seen how excessively minute would be the elastic forces developed in the production of distortion, in a rigid earth spheroid, commensurable with fluid tide-waves.[1]

In a paper "On the Secular Cooling of the Earth" (Trans. R.S.E., 1862, and Appendix to "Treatise, &c."), Sir W. Thomson applies a solution of Fourier to the determination of the interior temperature and its rate of increase downwards,

---

(1) See Additional Notes, p. 51.

[1] M. Delaunay, President of the French Academy, after quoting (Comptes rendus 1868) from the paper of Sir W. Thomson to which I have already referred, the results of Hopkins and some corroborating remarks from Sir W. Thomson's paper (referred to above), says: "Ainsi, on le voit, l'objection mise en avant par M. Hopkins, contre les idées généralement admises par les geologues sur la fluidité intérieure du globe terrestre, est regardée par plusieurs savants anglais comme parfaitement fondée. Je suis d'un avis diametralement opposé: je crois que l'objection de M. Hopkins ne repose sur aucun fondement réel." M. Delaunay then refers to an experiment made under his direction with a glass vase 0^{m} 24 in diameter, as furnishing decisive proof that the "viscosity" of a liquid as perfectly fluid as water even, is sufficient to cause it to take up the rotary motions of its enveloping shell, provided that those motions are relatively slow, as are those which constitute the precession and nutation of the earth; and he goes on to say: "Hence it does not appear to me possible to admit that the effect of the perturbing forces to which precession and nutation are due extend only to a portion of the mass of the terrestrial globe; the entire mass ought to be carried along (entrainée) by the perturbing actions, whatever may be the magnitude attributed to the interior fluid portion, and consequently the consideration of the phenomena of precession and nutation can furnish no datum for estimating the greater or less thickness of the solid crust of the globe."

M. Delaunay seems to be unaware that Sir W. Thomson coincides with Prof. Hopkins only in *this* (as the sequel of the very paper quoted shows), that he demands a great thickness of crust, and, moreover, that the interior, to the depth of this crust, shall be not merely "solid," but possessing a rigidity "several times as great as that of iron." I have endeavored to show that Sir W. Thomson's argument is irrefragable; but, based upon wholly different considerations, it is certain that no degree of "viscosity" assigned to an internal liquid will refute it.

I have remarked, at the outset of this discussion, that Prof. Hopkins' results "have not been generally accepted as decisive;" but I cannot admit that, as a test of their tenability, the experiment of M. Delaunay possesses the crucial character which he attributes to it. Viscosity, considered as an accelerating force tending to impart to a fluid the rotary motions of an enveloping shell, is directly proportional to the surface of contact, and inversely to the *mass* of contained liquid; in other words, it varies inversely as the diameter of the enveloping shell. The effect of viscosity of the fluid contents of the earth compared to those contained in a similar spherical envelope of only ten inches diameter, would be expressed (nearly enough) by the fraction $\frac{1}{40,000,000}$.

assuming a uniform primitive melting temperature of 7000° Fah., and a lapse of 100 millions of years since the cooling process commenced.

"The rate of increase of temperature from the surface downwards would be sensibly $\frac{1}{51}$ of a degree per foot for the first 100,000 feet or so. Below that depth the rate of increase per foot would begin to diminish sensibly. At 400,000 feet it would have diminished to about $\frac{1}{141}$ of a degree per foot. At 800,000 feet it would have diminished to less than $\frac{1}{50}$ of its initial value, that is to say, to less than $\frac{1}{2501}$ of a degree per foot; and so on, rapidly diminishing. Such is, on the whole, the most probable representation of the earth's present temperature, at depths of from 100 feet, where the annual variations cease to be sensible, to 100 miles, below which the whole mass, or all except a nucleus cool from the beginning, is (whether liquid or solid) probably at, or very nearly at, the proper melting temperature for the pressure at each depth."

The high rigidity demanded is difficult to conceive of in connection with a temperature in the solidified mass "at or near" that of melting, extending downwards indefinitely towards the centre of the earth. Hence Poisson's reasoning, which results in showing that the earth to have become thoroughly cooled and to have been subsequently reheated, *superficially*, harmonizes better with the demand for rigidity than that of Leibnitz, which supposes it to be now cooling from a (throughout) incandescent liquid state. In the latter case the law of actual temperature as deduced from Fourier's formulæ (as expressed above by Sir W. Thomson) would extend to the centre; in the former case only to the unmelted portion or to the "nucleus cool from the beginning."

Referring to the increase of temperature, with depth observed in mines, &c., Poisson remarks: "Fourier et ensuite Laplace ont attribué ce phénomène à la chaleur d'origine que la terre conserverait à l'époque actuelle et qui croîtrait en allant de la surface au centre, de tel sort qu'elle fût excessivement élévée vers le centre * * * en vertu de cette chaleur initiale la temperature serait aujourd'hui de plus de 2000 degrés à une distance de la surface égale seulement au centième du rayon; au centre elle surpasserait 200,000 degrés. * * * Mais quoique cette explication ait été généralement adoptée, j'ai exposé, dans mon ouvrage, les difficultés qu'elle presente, et qui m'ont paru la rendre inadmissible, * * * je crois avoir demontré que la chaleur developpée par la solidification de la terre a du se dissiper pendant la durée de ce phénomène, et que *depuis longtemps il n'en subsiste plus aucune trace.*" (Théorie de la Chaleur.)

Poisson, as is well known, attributes the increase of temperature, with depth, observed in the earth's crust, to the passage, at a remote period, of the solar system through hotter stellar regions, the temperature of which, he argues, should differ from place to place. Even a hypothetical case of the illustrious author, conforming to his theory, of an increased superficial temperature, 5000 centuries ago, of 200° C. diminishing by cooling (by transition to cooler stellar regions) to 5°, 500 centuries ago, and, subsequently, to the actual mean temperature, would scarcely meet the present demands for time, of palæontology; while the determinations of the conductivity of the earth's crust made near Edinburgh show, according to Sir W. Thomson, a necessity for increments of temperature of 25°, 50°, and 100°

See also p. 41 after p. 52

(Fah.), for past periods of only 1250, 5000, and 20,000 years, and authorize him to pronounce Poisson's hypothesis impossible, without destruction of life, or relegation of the date to so remote an era as to demand an intensely heated stellar region.

A reheating *after* solidification which should again fuse the surface to great depths would be, it seems to me, as "inadmissible" for the origin of observed subterranean temperatures, as the heat originally "developed by the solidification of the earth." Hence, the hypothesis of a reheating to fusion of the surface by impact of meteoric bodies would be likewise excluded by Poisson's theory of internal temperatures.

In the paper referred to (p. 39), Sir Wm. Thomson discusses the probable circumstances of solidification of the earth, assuming the known crust-materials (granite, &c.), in a molten state, as the constituent, and reasons that in consequence of the great condensation of granite in freezing, solidification must commence at the centre, and that "there could be no complete permanent incrustation all round the surface "till the globe is solid, with, possibly, the exception of irregular, comparatively "small spaces of liquid;" such separation of constituents in the process of crystallization taking place in all liquids composed of heterogeneous materials, and, indeed, is observable in the lava of modern volcanoes. He infers from the probable phenomena developed, into the discussion of which he goes at some length, "results sufficiently great and various to account for all that we see at present, and "all that we learn from geological investigation, of earthquakes, of upheavals, and "subsidences of solid, and of eruptions of melted rock."

Still we would, if possible, find reason to attribute a lower than "the proper melting temperature" to the solidified interior. Ice, indeed, preserves its rigidity unimpaired up to the point of fusion, and there may be a few other substances that have the like property; but it seems to be an exceptional one. The known constituents of the earth's crust certainly do not possess it, at least under ordinary pressures. If, as suggested by Prof. Joseph Le Conte (Am. Journal of Science, Nov. Dec. 1872), the "conductivity" be increased by pressure and *condensation*, such diminished temperatures may obtain.

NEW ADDENDUM.[1]

A FEW words are in place here concerning the results of the late Prof. Hopkins' investigation (against which M. Delaunay's objections, (note, page 38,) are especially directed), briefly stated, pages 34, 35. They are as follows: First FOR HOMOGENEOUSNESS. "Supposing the earth to consist of a homogeneous spheroidal shell (the ellipticities of the outer and inner surfaces being the same) filled with a fluid mass of the same uniform density as the shell;" then, "the precession will be the same, whatever be the thickness of the shell, as if the whole earth were homogeneous and solid."

SECOND, FOR HETEROGENEOUSNESS, his result may be thus expressed:

$$(a) \qquad P_1 - P' = P_1 \left\{ 1 - \frac{\varepsilon}{\varepsilon_1} \left(1 + \frac{s}{1 + \frac{h}{q^5 - 1}} \right) \right\};$$

"where P_1 denotes the precession of a solid homogeneous spheroid of which the ellipticity $=\varepsilon_1$, that of the earth's exterior surface, and P' the precession of the earth, supposing it to consist of an interior heterogeneous fluid contained in a heterogeneous spheroidal shell, of which the interior and exterior ellipticities are respectively ε and ε_1, the transition being immediate from the entire solidity of the shell to the perfect fluidity of the interior mass."

In the multiplier of $\frac{\varepsilon}{\varepsilon_1}$, second member of (a), q is the ratio of external to internal polar radius of the shell; s depends on the varying ellipticity and density of the strata of equal density of the shell; h depends on the density of the fluid interior. For a thin crust the coefficient in question is unity nearly; for a thick one it will be somewhat greater if ε be less than ε_1.

It cannot fail to be observed that, under the conditions just before expressed for *homogeneousness—i. e.*, equality of external and internal ellipticities—we get from the formula (s becoming zero) *the same result, i. e*, $P' = P_1$, as for that case.

In accordance with rational hypothesis as to the internal condition of the earth, equalities of ellipticities for the surfaces of a *thin* crust (and corresponding equality of densities), or closely approximate equalities would be expected. The necessity for a *thick* crust arises, therefore, from the alleged discrepancy between the observed and calculated annual precessions (50 seconds and 57 seconds), which, according to Prof. Hopkins, makes $\frac{P_1 - P'}{P_1} = \frac{1}{8}$, nearly, assuming the moon's mass $\frac{1}{70}$, and the earth's ellipticity $\frac{1}{300}$. (The real discrepancy is probably very much less. See page 34, *et sequentia*.)

[1] The original ADDENDUM, hurriedly written while the work was in the printer's hands, has been, in what follows, somewhat modified and amplified.

If, in applying the expression (a), the symbolic fraction, for a first approximation, be omitted, we have, according to above assumption of discrepancy, $\varepsilon = \frac{7}{8}\varepsilon_1$. This value of ε will be *in excess;* hence, the thickness of crust deduced from it will err the other way, and a determination on this basis will give a thickness which must, *in fact*, be exceeded.

The limit of solidity, proceeding inwards, may and probably does depend upon both temperature and pressure. Isothermal surfaces Prof. Hopkins finds to have *increasing* ellipticities. Surfaces of equal pressure, deduced from the hypothetical law of density, $A\frac{\sin qb}{b}$, have diminishing ellipticities, and if $qb_1 = 150°$ the above law agrees sufficiently well with the actual ellipticity and ratio of surface to mean density of the earth. This law for $\varepsilon = \frac{7}{8}\varepsilon_1$ demands a thickness of crust of $\frac{1}{4}$ the radius, or 1000 miles. This is a *minimum*, since the actual surface of solidification (lying between this and the corresponding isothermal surface) would have greater (and hence too great) ellipticity.

Before commenting upon this application, and upon the real meaning of the formula, I return to the case of homogeneousness. Some of the results arrived at by the analysis of Prof. Hopkins may be illustrated by the following considerations: The fluid spheroid, treated of p. 36, is subjected, by the attraction of the sun, to the distortion expressed by (47). This distortion, as shown by the form of the expression, is equivalent to an exceedingly slight rotational displacement[1] *of figure* about an equatorial axis, such as would be caused by displacing through a still more minute angle the planes of diurnal rotation. It is one of the beautiful results of the analysis to show that the change in the direction of the centrifugal force due to this slight obliquity of the planes of rotation is equivalent to turning forces at all points of the fluid exactly proportional to their distances from the equatorial axis.

Let now a rigid shell, exactly conforming internally to the external surface of the fluid, be applied, and the whole turned back until the planes of rotation are restored to perpendicularity to their axis; the precessional effect of the attracting body now operates upon the whole mass; for there are no longer counteracting tidal protuberances. If we take that part of (47) which is due to the *direct* action of the sun, viz., $\frac{3S}{r^3 g}\sin\theta\cos\theta\sin\lambda\cos\lambda\cos\varpi$ (for, the protuberances being repressed by the shell, the pressures on its interior which replace them will arise only from the direct action), and estimate it as a pressure and calculate the elementary couples for an internal ellipticity, e, we shall find the integral couple (and this corresponds with Prof. Hopkins' result) to be identical with (48) viz., exactly that due to the

[1] The required angle of the displacement is the height of the tidal wave (47), for $\varpi = 0$, divided by $\frac{dr}{d\lambda}$ for an ellipse of ellipticity, e, ($2e\sin\lambda\cos\lambda$). Prof. Hopkins shows that the corresponding divergence of the planes from perpendicularity develops a couple $= \frac{4}{15}\pi n^2 e$ multiplied by the sine of twice this arc (or twice the arc itself). Performing the operations we get, $\frac{24}{15}\pi e\frac{S}{r^3}\sin\theta\cos\theta$, in which we have the solar couple (19) and (48), which causes the displacement, since $e = \frac{15}{8\pi}(C - A.)$

couple which the sun would exert on the fluid mass considered as a solid.[1] It would increase the precessional force of the shell in the ratio $\frac{q^5}{q^5-1}$ of the analysis. By virtue of this pressure the fluid tends to transform its own precession into an augmented precession of the shell.

It requires, however, but an extremely minute angular separation of the axes of the shell and fluid to generate counter-pressures equivalent to those which caused the separation.[2] The divergence cannot, therefore, be progressive, but is simply a minute oscillation of the two axes, or a rotation around each other. In the latter form it appears in the analysis which, otherwise, gives to the internal fluid mass a precession *identical with that of the enveloping shell.*

Prof. Hopkins confines his analysis for the case of homogeneousness to equal ellipticities for the bounding surfaces of the shell. Excepting the case of sphericity for the inner surface, the result would be the same—viz., an unchanged precession, however the ellipticities might differ.

I now return to the formula (a) and remark, that it is an inaccurate expression for a slight difference (P_1—P') due to the fact that the spheroid is *heterogeneous*—that it is *not capable of being made a test of internal fluidity, or a measure of thickness of crust.*

I have already shown that for homogeneousness the couple due to pressure on the inner surface of the shell is identical with the sun-couple upon the fluid mass solidified, a result approximately true (as will be shown hereafter) if the density of the fluid strata vary. Hence, if we take the sum of the sun-couple exerted on a shell of interior and exterior ellipticities, ε and ε_1, and of the pressure-couple developed in the fluid,[3] and divide by the moment of inertia of the entire mass and by ω, we shall have the rate of gyration of the entire mass considered as a solid.

Referring to Prof. Hopkins' analysis and symbolism, the quotient will be[4]

$$(v) \qquad \frac{3}{2}\frac{\mu}{r^3\omega}\sin 2\Delta\,\frac{\int_a^{a_1}\rho'\frac{d(a^5\varepsilon')}{da'}da' + 2a^3\varepsilon\int_0^a \rho' a' da'}{\sigma(a_1) = \int_0^{a_1}\rho'\frac{da^5}{da'}da'}$$

[1] The lever arm is also $2e \sin\lambda \cos\lambda$. Multiply the above by this arm, by g, by the elementary surface $d\mu\, d\varpi$, and, again, by $\cos\varpi$, and we get the elementary component tending to tilt the shell. The integral, with proper substitutions, is equivalent again to (19) or (48).

[2] There is another process which may take effect in neutralizing internal pressure. I have remarked (last par. p. 6), that, considered as a perfectly rigid body, the precessional motions of the earth cannot be *precisely* those assumed. In fact, our imperfect integrals of the conditional differential equations present the anomaly of a varied motion in which the generating force *does no work;* no yielding to the tilting couple having place. There are necessarily some, too minute to be detected, nutational movements. In case the precessional force were augmented by so large a ratio as $\frac{q^5}{q^5-1}$ would be for a *thin* shell, *these* nutational movements would surpass in magnitude those necessary to generate the required counteracting pressures.

[3] I use provisionally Prof. Hopkins' computations for this, involving $\int_0^a \rho' a' da'$; its erroneousness will appear hereafter.

[4] The symbols μ, Δ, ω, correspond to S, θ, n, of p. 7; ρ' is the density of stratum, solid or fluid, for which ε' is the ellipticity, and a' the polar radius; a_1 is the external, and a the internal polar radius of the shell.

Denote the moment of inertia of the entire spheroid by $I_1 = \frac{8}{15}\pi\sigma(a_1)$

" " " " " " shell " $I' = \frac{8}{15}\pi[\sigma(a_1)-\sigma(a)]$

" " " " " " nucleus " $I'' = \frac{8}{15}\pi\sigma(a)$

Then $$\sigma(a_1) = \frac{I_1}{I'}\left(\sigma(a_1) - \sigma(a)\right)$$

and the above expression, reduced to precession, will become

$$(x) \qquad \frac{I'}{I_1}\left(1 + s + \frac{h}{q^5 - 1}\right) P \qquad (1)$$

Prof. Hopkins gets for the precession of the same spheroid considered as fluid within the shell, (his symbolic abbreviations used in both cases)

$$(y) \qquad \frac{(\gamma_2)}{(\gamma_1)+(\gamma_2)}\left(1 + s + \frac{h}{q^5 - 1}\right) P$$

In this last expression (γ_1) and (γ_2) denote coëfficients of gyration which one and the same couple (*i. e.* the centrifugal force, by pressure on the shell and by reaction on the fluid mass—*the assumption being made* that the latter, having its proportionate force on each particle, gyrates as a solid) produce upon the shell and fluid mass respectively. They should be therefore inversely proportional to the respective moments of inertia of the shell and nucleus, rendering the expressions (x) and (y) identical.

But this apparent identity is brought about by *assuming* that Prof. Hopkins' expression for the pressure-couple on the shell arising from the sun's attraction on the fluid to be identical (or at least approximately so) with that which would be exerted on the same heterogeneous fluid solidified; by which assumption I introduce in (x), $\frac{h}{q^5-1}$, $\left($which is Prof. Hopkins' symbolic abbreviation of

$$\frac{2a^3\int_0^a \rho' a' da'}{\sigma(a_1)-\sigma(a)}\Big)\text{: instead of } \frac{\int_0^a \rho' \frac{da^5\,\varepsilon'}{da} da'}{\varepsilon[\sigma(a_1)\,\sigma(a)]}$$

which latter expression belongs to the case just specified, of the *solidified* fluid.

Now in the case of nature—*i. e.* the earth with the received hypothetical laws of density and ellipticity, the two expressions differ in a ratio (about 4 : 3) so greatly exceeding unity, as to forbid the assumption of approximate equality. The error of the first expression will be better appreciated by referring to the quan-

---

[1] The interpretation of (x) and (y) is obvious. P is the coefficient of precession for a *homogeneous* shell of uniform ellipticity s; instead thereof let the shell be *heterogeneous* with same internal surface, but of an *external* ellipticity s_1. P for such a shell will have a fractional increment denoted by the ratio s. By the pressure of the internal fluid the precessional coefficient of this shell will be still further increased by a ratio *denoted* (a result of the analysis) by $\frac{h}{q^5-1}$. But the shell is constrained to carry along and to take up a common precession with the nucleus, and the coefficient will be thereby diminished in a ratio $\frac{I'}{I_1}$ of (x) or (according to Prof. Hopkins) the corresponding expression of (y). Since $P = P_1 \frac{s}{s_1}$, expression (a) is readily deducible from (y).

tities (γ_1) and (γ_2) in (y). The brief account given, p. 43, will show how, as resulting from the analysis, a rotating homogeneous fluid enveloped and confined by a shell reacts, with a *practical* rigidity conferred by rotation, against the shell, (when the respective axes of rotation are slightly separated) and thereby receives an angular motion "precisely as if it were solid," (Phil. Trans., 1839, p. 394). It is clear that, through this interaction, the shell, likewise, must receive an angular motion, and that these several angular motions must be in inverse ratio to the respective movements of inertia of shell and nucleus; and so, for homogeneousness, the analysis makes them. When we come to heterogeneousness, the same *modus operandi* is (and rightly) attributed to the fluid; and again, most clearly, the relative angular motions of shell and fluid should be in above-mentioned inverse ratio; whereas their ratio is quite differently computed to be $\frac{h}{q^5-1}$ of which the value has just been given. The error (for there is clearly one) is in the computation of the pressure-couple developed in the fluid and exerted upon the shell by centrifugal force, when their axes of rotation are slightly separated.

The same error of computation (exhibiting itself by the identical symbol, $\frac{h}{q^5-1}$) enters into the expression for the pressure-couple developed by solar attraction, and introduces *again* the above symbol as the third term of the second factor of (y).

Without going into lengthy discussion, it is sufficient to remark that both the centrifugal force and the foreign attraction produce in the strata of equal density of a heterogeneous fluid, *special configurations*, and expend themselves in so doing; and, moreover, that the prior establishment of these forms of equilibrium is assumed, and necessarily assumed, in the analysis. It is, therefore, unwarrantable to integrate (as is done) through the fluid mass these forces, as free forces, to get its pressure upon the shell.[1]

I have shown, I think, that in the expression (y) the first factor should be corrected to be, as it is in (x), $\frac{I'}{I_1}$.

The correction consists in substituting for $\frac{(\gamma_1)}{(\gamma_2)}$ in the *first* factor of (y), $\frac{\int_0^a \rho' \frac{da^5}{da} da'}{\sigma(a_1)-\sigma(a)}$ instead of $\frac{h}{q^5-1}$ [or its value, p. 45]. The *same* correction for $\frac{h}{q^5-1}$, introduced into the second factor, would require $\varepsilon \int_0^a \rho' \frac{da^5}{da} da'$ to be substituted for the second term of numerator of (v). But that (v) should belong to entire solidity we require $\int_0^a \rho \frac{d\, a^5 \varepsilon'}{da'} da'$.[2] Now these quantities differ inappreciably, for taking the entire

[1] It is obvious that the taking account of the internal motions by which the configurations produced by foreign attraction adapt themselves to the diurnal rotation, does not meet the point made.

[2] Although the errors in the two cases have the same expression, it does not follow that the corrections should be *identical*. The configurations of strata of like density due to foreign attraction are superinduced on the *previously established* configurations due to the centrifugal force, which have the varying ellipticity ε'. The correction should involve this ellipticity, and it is probable that the above symbol is, if we disregard the slight *internal motions*, the true one. Indeed, I think I may venture to affirm, that, given a heterogeneous fluid wholly enveloped by a rigid bounding surface,

integrals from zero to a_1 and multiplying by $\frac{8}{15}\pi$, the last becomes (Thomson and Tait, § 825),

$$\frac{2}{3} M a_1^2 (\varepsilon_1 - \tfrac{1}{2} m) = \mathfrak{C} - \mathfrak{A}, \text{ for the earth as actually constituted.}$$

And the first (deduced from Hopkins, Phil. Trans., 1840, pp. 203 and 204), is (nearly)

$$\frac{1}{3} M a_1^2 \varepsilon_1 = C - A, \text{ for same spheroid with } \textit{uniform} \text{ internal ellipticities.}$$

(M = mass of the earth).

[The value of the first, using the constants of density of Archdeacon Pratt, "Figure of the Earth," 4th ed., p. 113, is somewhat *less* than this last expression.] Now $m = \frac{1}{289}$ (ratio of centrifugal force to gravity) and hence $2(\varepsilon_1 - \frac{1}{2}m)$ is very little less than ε_1. For a fluid *nucleus*, the inequality would be still less.

Hence it appears that both (x) and (y) (when corrected) express very nearly the precession of the solidified earth; and, moreover, that the effect upon precession due to the variation of internal ellipticity is very small, the precession of the earth considered as rigid being, essentially, that corresponding to uniform ellipticity; or what is the same thing, that of a *homogeneous spheroid of its external form.*

This also appears in the comparison of the value of $\frac{C-A}{A}$, as established from observation, and the resulting *calculated* ellipticity. The first is .00327 and the second $\frac{1}{305}$ (Thomson and Tait, § 828). Now a homogeneous spheroid of the latter ellipticity would have for $\frac{C-A}{C}$ a value ($e - 4e^2$) of .00332; a difference of about $\frac{1}{66}$. Variations in the constants which enter into the expressions for internal density give rise to variations in the *calculated* ellipticity—and, of course, in the resulting precession; but if the *external* ellipticity is defined by a rigid shell, the effect of *internal* variation is, in the case in hand, almost *nil*. Hence, had the hypothetical consolidation of the earth, of p. 43, been carried to the very centre, no material approximation to the desired correction of $\frac{1}{8}$ in the calculated precession would have been found.[1] In fact, the problem for heterogeneousness

---

subjected to a foreign attraction, and a condition of static equilibrium assumed, the pressure-couple exerted by the fluid on the shell cannot differ from that which the attraction would exert on the solidified fluid.

In the case of homogeneousness, I have arrived (p. 43: the results, though based on an ellipticity corresponding to fluid equilibrium, hold good for any *small* ellipticity) at the exact expression for the pressure-couple, from the function expressing the tidal protuberance due to the foreign attraction. The tidal configuration of the heterogeneous earth, wholly liquefied, would result from the transcendental analysis of Hopkins, pp. 203, 204, or of Thomson and Tait, § 822–824, and the maximum height would be one foot, very nearly; but the pressure function cannot be readily deduced. It would depend on gravity and [§ 825] "the value of $C - A$ may be determined solely from a knowledge of surface or external gravity, or from the figure of the sea level without any data regarding the internal distribution of density."

[1] It is curious, to say the least, that there should be ground for the remark that the expressions (a) and (y), which latter, with its author's valuation of $\frac{(\gamma_1)}{(\gamma_2)}$, may be writen $\left\{1 + \frac{s}{1 + \frac{h}{q^5 - 1}}\right\} P$, give,

was practically solved under Prof. Hopkins' treatment of it when it was shown that, for homogeneousness, the precession for internal fluidity was the same as for solidity, and when it appeared that the analysis applied would exhibit the action of the heterogeneous fluid as if disposed in strata of like ellipticity with that of the inner shell surface. The erroneous computation of pressures has merely the effect of exaggerating in a like ratio the densities of all the fluid strata. Hence, and hence only, a resulting precession differing materially from that which would result from solidity.

In conclusion, I remark, 1st. The analysis of Prof. Hopkins, in its application to a homogeneous fluid and shell, seems to establish (and the result is confirmed by its harmony with tidal phenomena as developed in p. 43) that the rotation imparts to the fluid a *practical* rigidity[1] by which it reacts upon the shell as if it were

---

with decreasing internal ellipticities, for fluidity of nucleus *less precession than would belong to solidity*. This is obvious since his $\frac{h}{q^5-1}$ is greater than the ratio $\frac{I''}{I'}$ (nearly) which should take its place on the latter hypothesis.

[1] I do not concur with Sir William Thomson in the opinions quoted in note, p. 38, from Thomson and Tait, and expressed in his letter to Mr. G. Poulett Scrope ("Nature," February 1st, 1872), so far as regards fluidity, or imperfect rigidity, within an infinitely rigid envelope. I do not think the rate of precession would be affected.

That no increase arises from fluidity I have endeavored to show; and it is unquestionably a corollary of Prof. Hopkins' investigations. As regards imperfect rigidity, Sir William Thomson bases his argument upon the assumption that "the whole would not rotate as a rigid body round one 'instantaneous axis' at each instant, but the rotation would take place internally, round axes deviating from the axes of external figure, by angles to be measured in the plane through it and the line perpendicular to the ecliptic in the direction towards the latter line. These angular deviations would be greater and greater the more near we come to the earth's centre. * * * * * Hence the moment of momentum round the solsticial line would be sensibly less than if the whole mass rotated round the axis of figure."

If I do not misunderstand his language, Sir William Thomson assumes that the same *bending* distortion which would ensue from the application of a couple to the external portions of a non-rotating spheroid, would, equally and *identically*, take place in a rotating one: thus causing the angle made by the planes of the external rings of matter and the solsticial line to be increased; with a corresponding diminution of the component of *Cn* about this line.

In the case specified by him (an extreme one) while sensible and important nutational movements would ensue, the mean *precession* would be insensibly affected; but I do not think precisely *such* elastic yielding would take place.

As an extreme case of an infinitely rigid and infinitely *thin* shell containing matter completely destitute of rigidity, take the fluid spheroid of p. 36, and conceive it enveloped by such a shell. It is *still*, as shown, p. 43, susceptible (and susceptible *only*) of the extremely minute deflections of its planes of rotation by which precession is completely annihilated. Confer now upon the contents of the shell rigidity, uniform, or varying from surface to centre, continuously or discontinuously, in any arbitrary manner, and you have every possible case of imperfectly rigid matter contained within a perfectly rigid crust. I can attribute no other effect to the conferred rigidity than a restoration of the *lost* precession—in whole or in part; nor can I suppose the shell enveloping imperfectly rigid matter to change its obliquity *more* than that which contains the fluid; regard being had to conditions of *equilibrium* without reference to living forces generated.

I must remark that this hypothetical case, though as admissible for argument as any other form of "preternaturally rigid" crust, is exceptional. With a shell of *finite* moment of inertia, having some comparable relation to that of the fluid contents, the precession, instead of being annihilated, would be that due to the entire mass

a solid mass, while its pressure imparts to the shell the requisite couple to preserve the precession unchanged.

2d. The same practical rigidity is, with entire reason, attributed to the heterogeneous fluid by which (leaving out of view minute *relative* oscillations which do not affect the mean resultant in other natural phenomena and should not in this) the shell and fluid take a common precession.

3d. The two masses retaining their configurations, mutual relations, and rotary velocities, essentially unaltered by the hypothesis of internal fluidity, it would be a violation of fundamental mechanical principles were the resulting precession not identical with that due to the entire mass considered as solid.

4th. The common and identical precession of fluid and shell resulting from the analysis, is indispensable to any conception of *precession* for the earth as composed of thin shell and fluid; for otherwise internal equilibrium would be destroyed and the "figure of the earth" cease to have any assignable expression. The entire mass, fluid and solid must (without invoking the aid of "viscosity"), be "carried along in the precessional motion of the earth." The analysis I have examined demonstrates the possibility and exhibits the *rationale* of such a community of precession, but fails in the attempt to exhibit a test of the existence or absence of internal fluidity.

5th. The powerful pressures that would be exerted upon a *thin* and rigid shell would probably produce in it noticeable nutational movements;[1] while if the shell be *not* of a rigidity far surpassing that of the constituents of the cognizable crust, the "precessional motion of the earth" would, owing to the neutralizing effect of tidal protuberances, scarcely be observable.

---

[1] *Vide* p. 44, and note 2: without reference to conventional "Nutation" which is but a form of *precession*. In connection with these *relative* motions of shell and fluid, it is in place to allude to the "Vindication of Mr. Hopkins' method against the strictures of M. Delaunay," by the late Archdeacon Pratt ("Figure of the Earth," 4th ed., p. 132). He reasons, that, if at any moment, the crust and fluid be arranged as to density, "exactly as if they had been hitherto one solid mass and be moving alike, this state cannot possibly continue." For the shell will be acted upon not only by the foreign attraction but by the fluid pressure, and will "begin to move quicker," with a precession due to *both* the thence arising couples. That this should not occur requires, he estimates, the counteracting centrifugal force of a tidal protuberance, in a crust supposed 100 miles thick, of *seventy-four feet*.

The writer does not seem to be aware that the author whom he vindicates finds no such relative acceleration of the shell (*vide* p. 44, § 2, of this Addendum) as resulting from the pressure; and, strangely, for an authority on the "Figure of the Earth," fails to recognize that "an elevation of the outer surface of the crust"—that is a tidal distortion—of *a single foot*, would relieve the shell from all pressure. This is, perhaps, a natural result of the use of an expression (Prof. Hopkins') for the pressure which disregards the influence of "Figure."

ADDENDUM TO NOTE 1, PAGE 38.

THE apparent antagonism between the theorem of the text and that of Laplace suggests a few additional words. The theorem of Laplace is that "in whatever manner the waters of the ocean act upon the earth, either by their attraction, their pressure, their friction, or by the various resistances which they suffer, they communicate to the axis of the earth a motion which is *very nearly equal* to that it would acquire from the action of the sun and moon upon the sea, if it form a solid mass with the earth." (Méc. Cél., Bowditch [3345].)

The theorem is demonstrated in two distinct, quite different, manners. The last demonstration is founded upon the principle of the "conservation of areas;" and as the result of *this* demonstration the proposition is stated in the above quoted words.

The first demonstration is purely analytical, and, after stating that "this fluid" (*i. e.* of the ocean) "acts upon the terrestrial spheroid by its pressure and by its attraction," Laplace proceeds to find the analytical expressions for the precession and nutation-producing couples due to this pressure and to this attraction as they are modified by the attraction of the sun and moon upon the fluid. He then proceeds to calculate these couples for the material substance of the ocean, considered as rigidly connected (or forming a solid mass) with the earth. He finds the couples, so calculated, respectively, identical in the two cases, and epitomizes the result as follows: "the phenomena of the precession of the equinoxes and the nutation of the earth's axis *are exactly the same* as if the sea form a solid mass with the spheroid which it covers." [3287.]

But this demonstration is limited by the assumption that "the sea wholly covers the terrestrial spheroid or nucleus, that is of a regular depth, and suffers no resistance from the nucleus;" and *both* demonstrations imply an ocean of (relatively) small depth.

Under the last *mentioned* treatment of the subject the proposition of Laplace and that which I demonstrate are but the extreme phases exhibited by the solution of a problem, according as the datum be that the depth of the sea is minute (in which case its entire precession-producing couple, not effectively exerted upon its own mass, is *almost* wholly *transferred* to the solid nucleus); or that the nucleus is very small, in which case the lost precession-producing couple of the fluid is but in small part transferred to the nucleus—or wholly disappears with the vanishing of the latter.

I think, however, that the last-mentioned (first in point of order) demonstration of Laplace is not as general as the language quoted [3287] would indicate. The omission of variations of the radius-vector, R, in all the integrations gives rise to errors which do not seem to me to be identical in the two processes by which the couples are calculated, when the variation of depth is very small.

An apt illustration of the above remarks is derived from the supposition—as admissible as any other—that, the depth, γ, is constant. In this case, *whatever be the ellipticity* of the solid nucleus, the value of y (the height of the diurnal or precession-affecting tide, see [2253] [3333] and also p. 36) is zero, and, of course, the couples [3272] and [3273] become zero—as will be found by performing the integrations in those equations. So, of course, do expressions [3284] and [3285] become zero, with γ made constant. But these last *should not* be zero except for the case of sphericity of the nucleus.[1] The expressions do not seem to me capable of sustaining the inference which I have quoted [3287], when the depth of the sea is uniform; a case which most naturally presents itself to the mind.

---

[1] A shell of slight internal ellipticity and small *uniform* thickness has a precessional coëfficient of four-fifths the value of that of a shell bounded by surfaces of equal ellipticity. Hence *in general* the variations of R (or the ellipticity) produce effects insensible compared to those of the depth.

Mr. Airy (Tides and Waves, Art. 127) bases his demonstration of the theorem exclusively upon the principle of the conservation of areas, remarking at the outset, "if the earth and sea were so entirely disconnected that one of them could revolve for any length of time with any velocity, increasing or diminishing in any manner, while the other could revolve with any other velocity changing in any other manner, we could pronounce nothing as to the effect of the fluctuation" (tidal) "upon precession."

A spheroidal nucleus wholly covered by an ocean of uniform depth, suffering no resistance, does not seem to me to lack much for fulfilling the above conditions.

If *velocities* are generated in the waters of the ocean by solar (or lunar) attraction, the centrifugal forces due to them might be looked to (though not alluded to by Laplace) as agents for transferring, from the fluid to the nucleus, the precession-producing couples due to the fluid mass, especially in the above hypothetical case. It will be found, however, by reference to the expressions [2260], that they give rise to no couple, and are, moreover, very minute.

The motion which the displacements [2260] [2261] indicate is a slight oscillation of the axis of the fluid envelope, moving as a solid, about the axis of the nucleus, the angular distance between these axes being slightly less than 2 seconds: it is, I presume, that which a non-rotating shell would have were the attracting body, with constant distance and declination, to move, with angular velocity n, in right ascension. In the case in hand it is the fluid shell which revolves and, suffering no change of form, would be itself affected by its proper precessional couple to the exclusion of the oscillation above described.

ADDITIONAL NOTES.

Note to page 11.

(1) The process indicated is a more legitimate carrying out of the methods peculiar to this paper than what follows in the text. The tangent of MM' (35) may be (approx.) taken for the sine, and the cosine taken constant at unity, as may also be the cos I'.

From (32) we may calculate by developing and neglecting terms in which $\sin^2 I'$ enters

$$\sin i = (1 - \cos^2 i)^{\frac{1}{2}} = \sin I - \sin I' \cos I \cos n_3 t$$

$$\sin i \cos i = \sin I \cos I - \sin I' \cos 2I \cos n_3 t - \tfrac{1}{2} \sin^2 I' \sin 2I \cos^2 n_3 t$$

Introducing these values in (38) and (39), and integrating we get expressions identical with 44 and 45, except a (practically) immaterial difference in the coefficient of t in the first which becomes $1 - \frac{1}{2}\sin^2 I'$ instead of $1 - \frac{3}{2}\sin^2 I'$.

Note to page 24.

(1) The foregoing interpretation of the symbolic integral in (7), adopted with hesitation from authors cited, is based on assumed constancy of the angle ϕ; but this angle necessarily varies, slowly indeed, but progressively, by the azimuthal motion measured by $n \sin \lambda$. The conditions for the formation of a leminiscate are not, therefore, rigidly fulfilled. It will be found, however, taking into account a complete excursion, that the slight increment which will enure to the moment of the quantity of motion, $\sin^2\theta \frac{d\phi}{dt}$, from this cause on one side of the vertical, will be neutralized on the other, in consequence of the opposing signs of $\cos\phi$, in opposite azimuths; or, at least, the *resultant* increment or decrement will be a quantity of the second order in minuteness, and hence, affecting only in the same degree the azimuthal motion $\frac{d\phi}{dt}$.

Note to page 39.

(1) The differential attraction of the sun on any length $d\chi$ of the rod, at distance χ from the earth's centre is $\left(\frac{S}{(r-\chi)^2} - \frac{S}{r^2}\right) d\chi$, r being sun's distance. Integrate from $\chi = \chi$ to $\chi = R$ (the earth's radius).

(1) $$S\left(\frac{1}{r-R}-\frac{R}{r^2}-\frac{1}{r-x}+\frac{x}{r^2}\right)$$

The above divided by the coefficient of elasticity E will give the elongation per unit of length at any point. Multiply by dx and integrate from $x=0$ to $x=R$, and the total elongation is

$$\frac{S}{E}\left\{\frac{R}{r-R}-\frac{R^2}{2r^2}+\log\left(1-\frac{R}{r}\right)\right\}.$$

Since $\log\left(1-\frac{R}{r}\right)=-\frac{R}{r}-\frac{1}{2}\frac{R^2}{r^2}-\frac{1}{3}\frac{R^3}{r^3}-$ &c., the foregoing will reduce, approximately to $\frac{2}{3}\frac{S}{E}\frac{R^3}{r^3}$.

If $M=$ earth's mass, $g=$ gravity at its surface, and $\frac{1}{n}$ the ratio of M to S, and m the ratio of length of rod of weight E (per square inch section) to R, we shall have: $S=ngR^2$, $E=mgR$, and the above expression for total elongation becomes:

(2) $$\frac{2}{3}\frac{n}{m}R\frac{R^3}{r^3}$$

Take $\frac{R}{r}=\frac{4000}{92000000}=\frac{1}{23000}$; $R=4000\times5280$ feet; $n=316000$; $m=.472$ (the latter value based on $E=34$ mill'ns lbs. per square inch, and a steel rod of that section to weigh 3.4 lbs. per foot length) and the total elongation (2) becomes $0^{ft}.975$. The *maximum* extension per unit of length is at the centre, and is found by putting $x=0$ in (1) and dividing by E. It is $\frac{S}{E}\frac{R^2}{r^3}=\frac{n}{m}\frac{R^3}{r^3}=$ .000000055, indicating a strain of $1^{lb}.87$ per square inch. The ratio of the *total* elongation (2) to the total length of the rod R is two-thirds of the above, indicating *about* that ratio for the ellipticities of superficial and central strata of a steel globe distorted by the sun's attraction; a result thus rudely calculated which differs little from that given in Thomson and Tait, § 837.

(Fah.), for past periods of only 1250, 5000, and 20,000 years, and authorize him to pronounce Poisson's hypothesis impossible, without destruction of life, or relegation of the date to so remote an era as to demand an intensely heated stellar region.

A reheating *after* solidification which should again fuse the surface and extend to great depths, would be, it seems to me, as "inadmissible" for the origin of observed subterranean temperatures, as the heat originally "developed by the solidification of the earth." Of course the hypothesis of a reheating to fusion of the surface by impact of meteoric bodies would be likewise excluded by Poisson's theory of internal temperatures.

Hence, notwithstanding the high rigidity demanded, the foregoing determination of Sir Wm. Thomson seems to represent the most probable thermal condition of the earth's interior.

ADDENDUM. to no. 240, p. 41

THE note, written two years ago, to page 39, concerning M. Delaunay's experiment (then and since so often referred to as a conclusive refutation of the arguments of the opponents of the "thin shell" hypothesis) had for purpose—*not* to deny the reality of "viscosity" as a force which claimed attention in the discussion of the Hopkins problem (for I presume the notion of it has presented itself to every one who has looked over that analysis)—but to show the *un*-crucial character of the experiment. Just at this moment comes to my notice the letter of Sir Wm. Thomson to Mr. G. Poulett Scrope ("Nature," February 1st, 1872), in which what is known about "viscosity," and how great that of the fluid contents of the earth must be to produce the effect attributed to it by Delaunay, is set forth.

The remaining portion of the letter relates to the effects of imperfect rigidity of *the internal contents* upon precession, on the supposition of a "thin preternaturally rigid crust;" *i. e.*, in which the internal substance (*e. g.*, an "ordinary elastic jelly") is yielding, but for which the envelope is unchangeable in form; and are intended to show that the precession would be considerably more rapid than if the rigidity were infinite throughout.

A few additional words are in place here concerning the results of Prof. Hopkins' investigation (against which M. Delaunay's objections, note, page 38, are especially directed), briefly stated, pages 34, 35. They are as follows: First FOR HOMOGENEOUSNESS. "Supposing the earth to consist of a homogeneous spheroidal shell (the ellipticities of the outer and inner surfaces being the same) filled with a fluid mass of the same uniform density as the shell;" then, "the precession will be the same, whatever be the thickness of the shell, as if the whole earth were homogeneous and solid."

Second, FOR HETEROGENEOUSNESS, his result is thus expressed:

$$(a) \qquad P_1 - P^1 = \left(1 - \frac{\varepsilon}{\varepsilon_1}\right)\left\{1 - \frac{\eta}{1 + \frac{h}{q^5 - 1}}\right\} P_1$$

"where P_1 denotes the precession of a solid homogeneous spheroid of which the ellipticity $=\varepsilon_1$, that of the earth's exterior surface, and P^1 the precession of the earth, supposing it to consist of an interior heterogeneous fluid contained in a heterogeneous spheroidal shell, of which the interior and exterior ellipticities are respectively ε and ε_1, the transition being immediate from the entire solidity of the shell to the perfect fluidity of the interior mass."

In the second factor, second member of (a), q is the ratio of external to internal polar radius of the shell; η depends on the varying ellipticity and density of the strata of equal density of the shell; h depends on the density of the fluid interior. For a thin crust the factor in question is unity nearly; for a thick one it may be considerably less.

It cannot fail to be observed that, under the conditions just before expressed for *homogeneousness—i. e.*, equality of external and internal ellipticities—we get from the formula *the same result*, *i. e.*, $P^1 = P_1$, as for that case.

In our ignorance of the internal condition of the earth, equalities of ellipticities for the surfaces of a *thin* crust (and corresponding equality of densities), or closely approximate equalities, would be expected. The necessity for a *thick* crust arises, therefore, from the discrepancy between the observed and calculated annual precessions (50 seconds and 57 seconds), which, according to Prof. Hopkins, makes $\frac{P_1 - P^1}{P_1} = \frac{1}{8}$, nearly, assuming the moon's mass $\frac{1}{70}$ and the earth's ellipticity $\frac{1}{300}$. (The real discrepancy is probably less. See note, page 34.)

If, in applying the expression (a), the second factor, for a first approximation, is omitted, we have $\varepsilon = \frac{7}{8}\varepsilon_1$. This value of ε will be *in excess;* hence, the thickness of crust deduced from it will err the other way, and a determination on this basis will give a thickness which must, *in fact*, be exceeded.

The limit of solidity, proceeding inwards, may and probably does depend upon both temperature and pressure. Isothermal surfaces Prof. Hopkins finds to have *increasing* ellipticities. Surfaces of equal pressure, deduced from the hypothetical law of density $\frac{\sin qb}{b}$, have diminishing ellipticities, and if $qb_1 = 150°$ the above law agrees sufficiently well with the actual ellipticity and ratio of surface to mean density of the earth. This law for $\varepsilon = \frac{7}{8}\varepsilon_1$ demands a thickness of crust of $\frac{1}{4}$ the radius, or 1000 miles. This is a *minimum*, since the actual surface of solidification (lying between this and the corresponding isothermal surface) would have greater (and hence too great) ellipticity.

Before commenting upon this application, and upon the real meaning of the formula, I return to the case of homogeneousness. Some of the results arrived at by the analysis of Prof. Hopkins may be illustrated by the following considera-

tions: The fluid spheroid, treated of p. 36, is subjected, by the attraction of the sun, to the distortion expressed by (47). This distortion, as shown by the form of the expression, is equivalent to an exceedingly slight rotational displacement[1] *of figure* about an equatorial axis, such as would be caused by displacing through a still more minute angle the planes of diurnal rotation. It is one of the beautiful results of the analysis to show that the change in the direction of the centrifugal force due to this slight obliquity of the planes of rotation is equivalent to turning forces at all points of the fluid exactly proportional to their distances from the equatorial axis.

Let now a rigid shell, exactly conforming internally to the external surface of the fluid, be applied, and the whole turned back until the planes of rotation are restored to perpendicularity to their axis; the precessional effect of the attracting body now takes effect upon the whole mass; for there is no longer a neutralizing centrifugal force. If we take that part of (47) which is due to the *direct* action of the sun, viz., $\frac{3S}{r^3 g} \sin\theta \cos\theta \sin\lambda \cos\lambda \cos\varpi$ (for, the protuberances being repressed by the shell, the pressures on its interior which replace them will arise only from the direct action), and estimate it as a pressure and calculate the elementary couples for an internal ellipticity, e, we shall find the integral couple (and this corresponds with Prof. Hopkins' result) to be identical with (48) viz., exactly that due to the couple which the sun would exert on the fluid mass considered as a solid.[2] It would increase the precessional force of the shell in the ratio $\frac{q^5}{q^5-1}$ of the analysis. By virtue of this pressure the fluid tends to transform its own precession into an augmented precession of the shell.

It requires, however, but an extremely minute angular separation of the axes of the shell and fluid to generate counter-pressures equivalent to those which caused the separation.[3] The divergence cannot, therefore, be progressive, but is simply a minute oscillation of the two axes, or a rotation around each other. In the latter

---

[1] The required angle of this displacement is the height of the tidal wave (47), for $\varpi = 0$, divided by $\frac{dr}{d\lambda}$ for an ellipse of ellipticity, e, ($2\,e \sin\lambda \cos\lambda$). Prof. Hopkins shows that such a divergence from perpendicularity develops a couple $= \frac{4}{15}\pi n^2 e$ multiplied by the sine of twice this arc (or twice the arc itself). Performing the operations we get, $\frac{24}{15}\pi e \frac{S}{r^3} \sin\theta \cos\theta$, in which we have the solar couple (19) and (48), which causes the displacement, since $e = \frac{15}{8\pi}(C-A)$.

[2] The lever arm is also $2\,e \sin\lambda \cos\lambda$. Multiply the above by this arm, by g, by the elementary surface, $d\mu\, d\varpi$, and, again, by $\cos\varpi$, and we get the elementary component tending to tilt the shell. The integral, with proper substitutions, is equivalent again to (19) or (48).

[3] There is another process which may take effect in neutralizing internal pressure. I have remarked (last par. p. 6), that, considered as a perfectly rigid body, the precessional motions of the earth cannot be *precisely* those assumed. In fact, our imperfect integrals of the conditional differential equations present the anomaly of a varied motion in which the generating force *does no work;* no yielding to the tilting couple having place. There are necessarily some, too minute to be detected,

form it appears in the analysis which, otherwise, gives to the internal fluid mass a precession *identical with that of the enveloping shell;* an important fact to be noted.

Prof. Hopkins confines his analysis for the case of homogeneousness to equal ellipticities for the bounding surfaces of the shell. Excepting the case of sphericity for the inner surface, the result would be the same—viz., an unchanged precession, however the ellipticities might differ.

I now return to the formula (*a*) and remark, that it is an approximate expression for the difference $(P_1 - P^1)$ due to the fact that the spheroid is *heterogeneous* —that it is *not capable of being made a test of internal fluidity, or a measure of thickness of crust.*

I have already shown that for homogeneousness the couple due to pressure on the inner surface of the shell is identical with the sun-couple upon the fluid mass —a result approximately true if the density of the fluid strata vary, the methods by which pressures are obtained in the latter case only differing by the introduction of the variable density into the differentials to be integrated. Hence, if we take the sum of the sun-couples exerted on a shell of interior and exterior ellipticities, ε and ε_1, and upon the fluid nucleus considered as solid, and divide by the moment of inertia of the entire mass and by ω, we shall have the rate of precession of the entire mass considered as a solid.

Referring to Prof. Hopkins' analysis and symbolism, the quotient will be

$$\frac{3}{2}\frac{\mu}{r^3\omega}\sin 2\Delta\,\frac{\int_a^{a_1}\rho\frac{d(a^5\varepsilon')}{da'}da' + 2f(a)a^3\varepsilon}{\sigma(a_1)} \qquad (1)$$

Denote the moment of inertia of the entire spheroid by I_1
" " " " " " shell " I

Then $$\sigma(a_1) = \frac{I_1}{I}\left(\sigma(a_1) - \sigma(a)\right)$$

and the above expression will reduce to

$$(x) \qquad \frac{\iota}{I_1}\left(1 + s + \frac{h}{q^5-1}\right) A\,\frac{q^5-1}{q^5}$$

---

nutational movements. In case the precessional force were augmented by so large a ratio as $\frac{q^5}{q^5-1}$ would be for a *thin* shell, *these* nutational movements would surpass in magnitude those necessary to generate the required counteracting pressures.

[1] This implies, of course, that the second term of the numerator should be equivalent to the symbolic expression $\int_o^a \rho\frac{d(a^5\varepsilon)}{da}da$, which would be used for the couple due to the contents of the shell regarded as solid. The statement of the text is, however, demonstrably true. The difference between the couple due to *pressure* (used above from Prof. Hopkins' determination) and that denoted by the symbol just written, can only be due to the portion of the sun-force expended in producing *internal motions* in the fluid, proved in the analysis to be, for the case in hand, omissible. Hence, the *analytical* discrepancy depends upon such neglected quantities: but, more generally, the effect of such motions would be practically *nil* in a rotating body.

Prof. Hopkins gets for the precession of the same spheroid considered as fluid within the shell

$$(y) \qquad \frac{(\gamma_2)}{(\gamma_1)+(\gamma_2)}\left(1+s+\frac{h}{q^5-1}\right)A\frac{q^5-1}{q^5}$$

In this last expression (γ_1) and (γ_2) denote coëfficients of gyration which one and the same couple (*i. e.* the centrifugal force, by pressure on the shell and by reaction on the fluid mass—*the assumption being made* that the latter, having its proportionate force on each particle, gyrates as a solid) produce upon the shell and fluid mass respectively. They are therefore inversely proportional to the respective moments of inertia of the shell and nucleus;[1] hence, the expressions (x) and (y) are identical. The precession is therefore the same whether the entire spheroid be solid, or whether it be a shell with a fluid nucleus.[2]

If confirmation be needed it will be found in the fact that the analysis itself gives, as in the case of homogeneousness, the same precession to the internal fluid spheroid as to the shell.

The two masses retaining their configurations, dispositions of matter, and rotary velocities, essentially unaltered by the hypothesis of internal fluidity, it would be a violation of fundamental mechanical principles were the resulting precession not identical with that due to the entire mass considered as a solid.[3]

I am aware that the author's ratio, $\frac{h}{q^5-1}$, is not identical with the above—but it evidently should be. The discrepancy is of the same character as that treated in the note to p. 44 concerning the value of h for the sun-couple; its origin being, unquestionably, in the assumption upon which the value of (γ_2) is independently calculated, viz.: that the planes of rotation are, by the angular separation of the axes of the shell and fluid, deflected in *perfect planes*. This assumption (which does not at all affect the calculated value of (γ_1)), is, as we have seen (p. 43 and note), sustained for the case of homogeneousness. It is not so in the other case. The internal denser fluid rings undergo less deflection than the lighter ones. It must be borne in mind that the angle ϵ, of these deflections, uniform or not, is very minute, even with reference to the angular separation of the axes, always to be regarded as extremely small; hence of the second order of minuteness.

[2] An exception must be made for perfect, or nearly perfect, sphericity of interior. In the expression (y), s becomes infinite and A zero for this case.

[3] The *oneness* of the precession of the shell and fluid contents, resulting from the analysis, may be put in a stronger light by the statement that for the sun or moon, *separately*, a suitable determination of arbitrary constants will eliminate even the minute oscillation about each other of the axes (pp. 43, 44), of shell and fluid, which depends on (γ_1) and (γ_2); this unison, I think, could not really obtain; but it is quite consistent with the convenient but imperfect view of gyration which makes it *only* a motion perpendicular to the plane of the generating couple.

I do not concur with Sir William Thomson in the opinions quoted in note, p. 38, from Thomson and Tait, and expressed in his letter to Mr. G. Poulett Scrope, so far as regards fluidity, or imperfect rigidity, within an infinitely rigid envelope. I do not think the rate of precession would be affected.

That no increase arises from fluidity I have endeavored to show; and it is, unquestionably, a corollary of Prof. Hopkins' investigations. As regards imperfect rigidity, Sir William Thomson bases his argument upon the assumption that "the whole would not rotate as a rigid body round one 'instantaneous axis' at each instant, but the rotation would take place internally, round axes deviating from the axis of external figure, by angles to be measured in the plane through it and the line perpendicular to the ecliptic in the direction towards the latter line. These angular deviations would be greater and greater the more near we come to the earth's centre. * * * * * Hence the

Of course, the practical application (briefly sketched at p. 42) of Prof. Hopkins' result to the case of the earth should amount only to this—that, so great a difference as $\frac{1}{8}$ between calculated and observed precession is not in accordance with *probable* laws of density. On the other hand, Thomson and Tait show, § 828, that for entire solidity, the observed precession is quite consistent with the actual mean density and probable (Laplace's) law of density.

In conclusion, it has been shown, that according to the celebrated investigation which I have examined, the entire mass of the spheroid, fluid and solid, *is* "carried along in the precessional motion of the earth;" and under that point of view only, (regarding the crust as rigid) I incline to the opinion of Mr. Delaunay that "the consideration of the phenomena of precession and nutation can furnish no datum for estimating the greater or less thickness of the solid crust of the globe."

---

moment of momentum round the solsticial line would be sensibly less than if the whole mass rotated round the axis of figure."

If I do not misunderstand his language, Sir William Thomson assumes that the same *bending* distortion which would ensue from the application of a couple to the external portions of a non-rotating spheroid, would, equally and *identically*, take place in a rotating one: thus causing the angle made by the planes of the external rings of matter and the solsticial line to be increased; with a corresponding diminution of the component of *Cn* about this line.

In the case specified by him (an extreme one) while sensible and important nutational movements would ensue, the mean precession would be very slightly affected; but I do not think precisely *such* elastic yielding would take place.

As an extreme case of an infinitely rigid and infinitely *thin* shell containing matter completely destitute of rigidity, take the fluid spheroid of p. 36, and conceive it enveloped by such a shell. It is *still*, as shown, p. 43, susceptible (and susceptible *only*) of the extremely minute deflections of its planes of rotation by which precession is completely annihilated. Confer now upon the contents of the shell rigidity, uniform, or varying from surface to centre, continuously or discontinuously, in any arbitrary manner, and you have every possible case of imperfectly rigid matter contained within a perfectly rigid crust. I can attribute no other effect to the conferred rigidity than a restoration of the *lost* precession—in whole or in part; nor can I suppose the shell enveloping imperfectly rigid matter to change its obliquity *more* than that which contains the fluid; regard being had to conditions of *equilibrium* without reference to living forces generated.

I must remark that this hypothetical case, though as admissible for argument as any other form of "preternaturally rigid" crust, is exceptional. With a shell of *finite* moment of inertia, having some comparable relation to that of the fluid contents, the precession, instead of being annihilated, would be that due to the entire mass.

PUBLISHED BY THE SMITHSONIAN INSTITUTION,

MAY, 1872.

ADDENDUM TO NOTE 1, PAGE 38.

THE apparent antagonism between the theorem of the text and that of Laplace suggests a few additional words. The theorem of Laplace is that "in whatever manner the waters of the ocean act upon the earth, either by their attraction, their pressure, their friction, or by the various resistances which they suffer, they communicate to the axis of the earth a motion which is *very nearly equal* to that it would acquire from the action of the sun and moon upon the sea, if it form a solid mass with the earth." (Méc. Cél., Bowditch [3345.])

The theorem is demonstrated in two distinct, quite different, manners. The last demonstration is founded upon the principle of the "conservation of areas;" and as the result of *this* demonstration the proposition is stated in the above quoted words.

The first demonstration is purely analytical, and, after stating that "this fluid" (*i. e.* of the ocean) "acts upon the terrestrial spheroid by its pressure and by its attraction," Laplace proceeds to find the analytical expressions for the precession and nutation-producing couples due to this pressure and to this attraction as they are modified by the attraction of the sun and moon upon the fluid. He then proceeds to calculate these couples for the material substance of the ocean, considered as rigidly connected (or forming a solid mass) with the earth. He finds the couples, so calculated, respectively, identical in the two cases, and epitomizes the result as follows: "the phenomena of the precession of the equinoxes and the nutation of the earth's axis, *are exactly the same* as if the sea form a solid mass with the spheroid which it covers." [3287.]

But this demonstration is limited by the assumption that "the sea wholly covers the terrestrial spheroid or nucleus, that is of a regular depth, and suffers no resistance from the nucleus;" and *both* demonstrations imply an ocean of (relatively) small depth.

Under the last *mentioned* treatment of the subject the proposition of Laplace and that which I demonstrate are but the extreme phases exhibited by the solution of a problem, according as the datum be that the depth of the sea is minute (in which case its entire precession-producing couple, not effectively exerted upon its own mass, is *almost* wholly *transferred* to the solid nucleus); or that the nucleus is very small, in which case the lost precession-producing couple of the fluid is but in small part transferred to the nucleus—or wholly disappears with the vanishing of the latter.

I think, however, that the last-mentioned (first in point of order) demonstration of Laplace is not as general as the language quoted [3287] would indicate. The omission of variations of the radius-vector, R, in all the integrations gives rise to errors which do not seem to me to be identical in the two processes by which the couples are calculated; and in the second calculation the errors thus involved are commensurable, when the variation of depth is not great, with the quantities sought. In expressions [3284] [3285] all manifestation of *the ellipticity* of the inner surface of the solid shell is lacking; yet upon *this* ellipticity, as much as upon the variation of the shell thickness γ, the couples depend, when the latter variation is relatively small.

An apt illustration of the above remarks is derived from the supposition—as admissible as any other—that γ is constant. In this case, *whatever be the ellipticity* of the solid nucleus, the value of y (the height of the diurnal or precession-affecting tide, see [2253] [3333] and also p. 36) is zero, and, of course, the couples [3272] and [3273] become zero—as will be found by performing the integrations in those equations. So, of course, do expressions [3284] and [3285] become zero, with γ made constant. But these last *should not* be zero except for the case of sphericity of the nucleus.[1] The expressions do not seem to me capable of giving with sufficient accuracy the general values of the couples, or of sustaining the inference which I have quoted [3287], even though otherwise tenable

[1] A shell of slight internal ellipticity and small *uniform* thickness has a precessional coëfficient of half the value of that of a shell bounded by surfaces of equal ellipticity.

Mr. Airy (Tides and Waves, Art. 127) bases his demonstration of the theorem exclusively upon the principle of the conservation of areas, remarking at the outset, "if the earth and sea were so entirely disconnected that one of them could revolve for any length of time with any velocity, increasing or diminishing in any manner, while the other could revolve with any other velocity changing in any other manner, we could pronounce nothing as to the effect of the fluctuation" (tidal) "upon precession."

A spheroidal nucleus wholly covered by an ocean of regular depth, suffering no resistance, does not seem to me to lack much for fulfilling the above conditions; especially if the "regular depth" be *constant*.

If *velocities* are generated in the waters of the ocean by solar (or lunar) attraction, the centrifugal forces due to them might be looked to (though not alluded to by Laplace) as agents for transferring, from the fluid to the nucleus, the precession-producing couples due to the fluid mass, especially in the above hypothetical case. It will be found, however, by reference to the expressions [2260], that they give rise to no couple, and are, moreover, very minute.

The motion which the displacements [2260] [2261] indicate is a slight oscillation of the axis of the fluid envelope, moving as a solid, about the axis of the nucleus, the angular distance between these axes being slightly less than 2 seconds: it is, I presume, that which a non-rotating shell would have were the attracting body, with constant distance and declination, to move, with angular velocity n, in right ascension. In the case in hand it is the fluid shell which revolves and, suffering no sensible change of form, would be itself affected by its proper precessional couple.

14 DAY USE
RETURN TO DESK FROM WHICH BORROWED
LOAN DEPT.
RENEWALS ONLY—TEL. NO. 642-3405
This book is due on the last date stamped below, or on the date to which renewed.
Renewed books are subject to immediate recall.

RECEIVED
SEP - 9 '69 - 12 AM
LOAN DEPT.
JUL 27 1972 9
Aug. 27, 72
Sept 27, 72
Oct 27, 72
REC'D LD OCT 28 '72 -1 PM 35
JUL 30 1973 7
REC'D CIRC DEPT
DEC 26 1975
DEC 8 '75
REC. CIR. DEC 8 '75
PHOTOCOPY JAN 26 '90

INTERLIBRARY LOAN
JAN 9 1990
UNIV. OF CALIF., BERK.

LD21A-60m-6,'69
(J9096s10)476-A-32
General Library
University of California
Berkeley

U. C. BERKELEY LIBRARIES
C054910896

8947
Q11
S68
v.19

UNIVERSITY OF CALIFORNIA LIBRARY